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SEQUENTIAL SPATIAL EQUILIBRIA UNDER INCOMPLETE INFORMATION

by

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RESUME

Nous considérons un modèle à deux étapes de concurrence spatiale duopolistique : choix séquentiel de localisation sur un segment (étape 1) et choix simultané (Bertrand) de prix livrés (étape 2). A l'étape 1, l'information du second joueur sur le coût marginal (élevé ou faible) du premier est incomplète. Ainsi, nous obtenons un modèle classique de signaux dans lequel apparaissent des contraintes naturelles sur l'espace des signaux possibles. L'objet de l'analyse est d'identifier et de caractériser les équilibres bayesiens parfaits satisfaisant divers critères de raffinement (le critère intuitif et le critère D1) lorsque les contraintes sur les signaux sont serrées. Le principal résultat est que l'équilibre séparateur n'est plus le seul équilibre robuste au critère D1; et les équilibres robustes deviennent alors sensibles aux croyances a priori des joueurs. Pour un intervalle de localisation donné, plus la différence de coût entre les deux types de premier joueur est grande, plus souvent les contraintes seront serrées. La caractérisation du rôle de ces contraintes dans les modèles de signaux repose sur une description systématique de tous les équilibres et de leur dépendance sur les croyances a priori.

Mots clés : Dupole spatial, équilibre bayesien, raffinements.

ABSTRACT

We study a model of spatial duopoly competition with sequential entry on a segment in the choice of locations (stage 1 of the game) and then simultaneous Bertrand competition in delivered prices (stage 2). The second mover at stage one has incomplete information on the marginal cost (high or low) of the first mover. Hence, we obtain a classical model of signalling in which there exist natural constraints on the range of possible signals. The focus of the analysis is the identification and characterization of perfect Bayesian equilibria robust to various refinement criteria (the intuitive criterion and D1) when the constraints on signalling are tight. The main result is that the separating equilibrium is no longer the only equilibrium surviving the D1 criterion. An important consequence of this result is that the robust equilibria become highly sensitive to the prior beliefs of the players. For a given interval of locations, the higher the cost difference between the low type and the high type of the first mover, the more often the constraint will be tight. The characterization of the role of these constraints in signalling models requires a systematic description of all the equilibria and their dependence on prior beliefs.

Key words : Spatial Duopoly, Bayesian Equilibrium, Refinements.

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1. INTRODUCTION

In his famous address before the American Mathematical Society in April 1928, Hotelling(1929) mentionned that "After the work of the late professor F.Y. Edgeworth one may doubt that anything further can be said on the theory of competition among a small number of entrepreneurs. However ... "(p. 41) and he went on with the development of his classic paper on a spatial representation of duopoly competition in differentiated products. More than 50 years later, Eaton and Lipsey (1980) wrote "In this paper we examine the use of product-specific capital goods as vehicles for entry-deterring *commitments*. It may seem surprising that there is anything left to say on this subject in view of such works as ... . However..." (p. 721). In spite of the fact that one may think everything has been written and said about spatial (or product-specification) competition, it seems to us that one important feature of real world competition has been left out or at least neglected so far, namely the incompleteness of the information structures characterizing decision-makers in spatial competition. It is our objective in this paper to analyse one particular source of incomplete information pertaining to one entrepreneur's production conditions, namely costs, which are unobserved by the competitor.

We consider a simple spatial framework where the space is represented by the interval [0,1]. There is a continuum of consumers distributed uniformly on [0, 1] with a density of 1. Each consumer has a reservation price r and consumes one unit of some good when its price is no greater than r. There are two firms, each with one single plant. In stage 1, the firms choose the locations of their plants in a sequential fashion, with firm 1 being the first-mover. The firms cannot relocate thereafter, that is we assume that capital is irreversible, location specific or durable enough to make exit prohibitively costly. In stage 2, the firms compete in delivered prices in a Bertrand-Nash (simultaneous move) fashion(1). Both firms produce at constant marginal costs, transport costs are proportional (equal) to the distance travelled and there are no sunk costs(2). Entry by a third firm is blockaded.

We will consider that the production cost of firm 1 is commonly known to be either c or $c - \alpha$ with some probabilities (prior beliefs) while the production cost of firm 2 is commonly known to be c. We will assume that, before choosing its location, firm 1 knows whether its production cost is c or $c - \alpha$. Firm 2 observes only the location of firm 1 before deciding on its own location. For simplicity we assume that in stage 2 firm 2 has learned firm 1's type before Bertrand competition takes place.

Our model is a monotonic signaling game as defined by Cho and Sobel (1988) (see section 7) for which there exists only one Bayesian perfect equilibrium surviving the D1 criterion of "divinity" (implying in the present case "universal divinity" and the "never a weak response" criteria). We will characterize the kind of D1 equilibrium arising under all possible prior beliefs and cost differences α. The choice of location by firm 1 may transmit at equilibrium the value of its marginal cost and it is well known that in signaling models, when a condition of single crossing is satisfied (as it is the case in the present model), only the separating equilibria survive the D1 criterion if such equilibria exist. We will show in section 4 that when the cost difference α is below some upper bound the only D1 equilibrium is a pure strategy separating equilibrium and that for higher values of α no separating equilibrium exists. This result may seem at first sight surprising because it might have been expected that for high cost differences it is more difficult for the high cost firm to be pooled with the low cost one. But in a spatial competition context or a finite linear maket the location gap between the two types of the firm is bounded for natural economic reasons, hence also the penalty for the high cost firm trying to hide its type[3]is also bounded above. So for relatively low discrepancies in the costs of the two types of firm 1, prior beliefs do not play any role in the determination of the nature of equilibrium. On the contrary, for higher values of α the prior beliefs become crucial in this determination. For priors putting a sufficiently high weight on the low cost firm 1, the only D1 equilibrium is a pure strategy pooling equilibrium at the center of the market as shown in section 5. The bound on the probability of the low cost firm 1 over which only this type of equilibrium is surviving D1 is a decreasing function of α. For priors under this bound and the same set of values of α we have only semi-separating equilibria where the low cost firm 1 is choosing to locate at the center of the market and the high cost one is mixing between the center of the market and its profit maximizing location under full information (see section 6).

In the next section we analyse the complete information case. With section 3 we begin the study of the incomplete information case by presenting definitions and preliminary results ; we then analyse the separating equilibria (section 4), the pooling equilibria (section 5) and the semi-separating equilibria (section 6). In section 7 we make use of the results of Cho and Sobel (1988) to show how the space of (prior beliefs, technological parameter) can be partitioned with respect to the respective types of equilibria.

2. THE COMPLETE INFORMATION CASE

Let $p_i(x ; x_1 , x_2)$ be the price quoted by firm i to a consumer located at x when the firms are located respectively at x_1 and x_2 on the [0,1] interval. Lederer & Hurter (1986) refer to such price schedules as discriminatory pricing through the absorption of transportation costs. They claim that it is typical of goods with low value relative to transportation costs in markets characterized by oligopolistic competition and low price elasticity of demand, such as cement, plywood, fertilizer and sugar. Competition in price schedules was also discussed among others by Hoover (1937), Lederer (1981), Carlton (1983) and Hobbs (1986) and by Thisse and Vives (1988).

Let us assume that if a consumer is quoted the same price by both firms, then he buys from the firm which realizes on his demand the larger profit ; if both firms realize the same profits on that sale, then he buys from either one with probability 1/2. Under this indifference rule assumption, one can show that the following pricing schedule strategy is a Bertrand-Nash equilibrium for the second stage of the game, that is for given locations x_1 and x_2 [see Hurter & Lederer (1985) and Frayssé & Grimaud (1990)] when $c_1 = c_2 = c$.

$$(1) \qquad p_1 = p_2 = p(x ; x_1 , x_2) = c + \text{Max} \{ | x - x_1 | , | x - x_2 | \}$$

For this equilibrium, the market is split into two areas. Firm 1 will sell at $p(x ; x_1 , x_2)$ to those consumers located at x for $x \leqslant (x_1 + x_2) /2$ and firm 2 will sell at $p(x ; x_1 , x_2)$ to those consumers located at x for $x \geqslant (x_1 + x_2)/2$. To verify that (1) is indeed a Bertrand-Nash equilibrium, it suffices to check that p_2 is a best-reply to p_1. Consider a consumer located at some $x \in [0, (x_1 + x_2)/2]$. The price $p_2(x ; x_1 , x_2) = c + x_2 - x$ is the best-reply price which firm 2 can choose since for $p_2 > c + x_2 - x$ it sells nothing and for $p_2 < c + x_2 - x$ it does not cover the sum of its production and delivery costs. For a consumer located at $x \in [(x_1 + x_2)/2 , 1]$ again the price $p_2(x ; x_1 , x_2) = c + x - x_1$ is the best-reply price for firm 2 since for $p_2 > c + x - x_1$ it sells nothing and for $p_2 < c + x - x_1$ it sells and covers its production and delivery costs but abandons a revenue equal to $c + x - x_1 - p_2$. Hence (1) is a Bertrand-Nash equilibrium for the second stage of the game. It is interesting to note that the price schedule $p(x ; x_1 , x_2)$ defined by (1) is such that consumers located closer to a store may be charged higher prices than consumers located further away from the store.

Given x_1 , x_2 and (1), the profit PR_i of firm i can be expressed as a function of locations x_1 and x_2 :

$$(2.1) \qquad PR_1(x_1, x_2) = \int_0^{(x_1+x_2)/2} \{[c + x_2 - x] - [c + |x - x_1|]\}\, dx$$

$$= (x_2 - x_1)(3x_1 + x_2)/4$$

$$(2.2) \qquad PR_2(x_1, x_2) = \int_{(x_1+x_2)/2}^{1} \{[c + x - x_1] - [c + |x - x_2|]\}\, dx$$

$$= (x_2 - x_1) - (x_2 - x_1)(x_1 + 3x_2)/4$$

Those profit levels obtained in the second stage as functions of locations chosen in stage 1 can be used to solve for optimal location at stage 1 in a subgame perfect equilibrium.

Given that in stage 1 (long run) the firms choose their respective locations sequentially with firm 1 as leader, we must first characterize the location choice of firm 2 as a function of the location of firm 1. The profit function $PR_2(x_1, x_2)$ is concave in x_2 for a given x_1 . Maximizing $PR_2(x_1, x_2)$ with respect to x_2 leads to

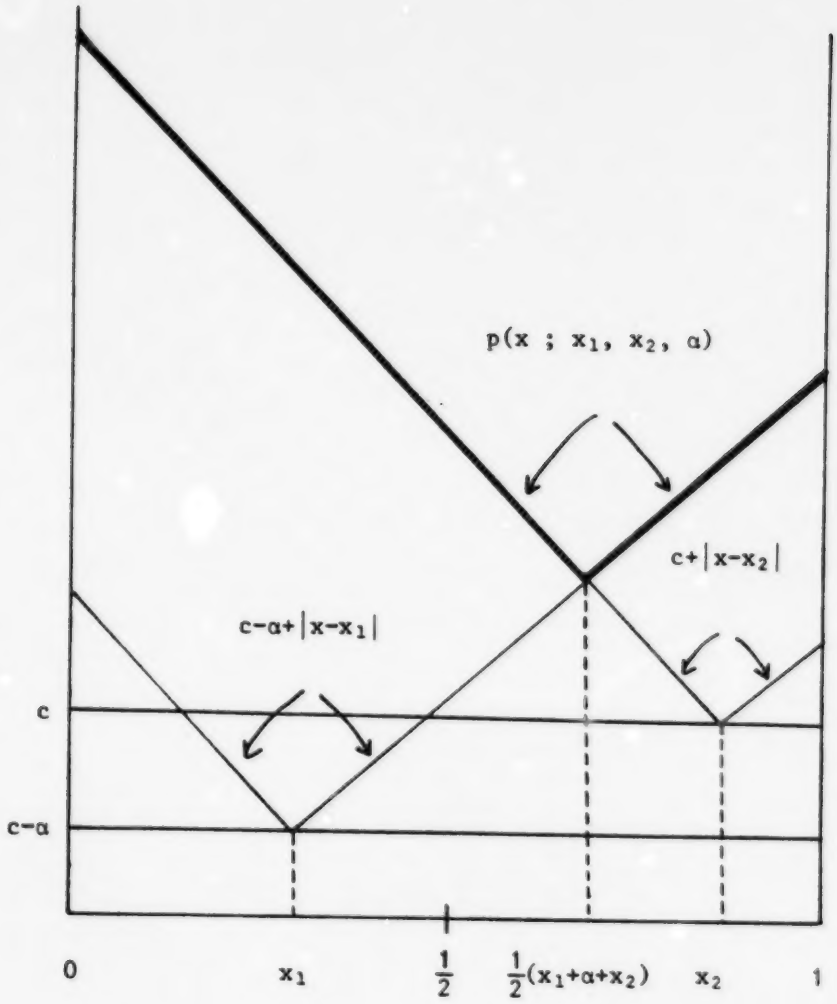
$$(3) \qquad x_2^* = x_2(x_1) = (2 + x_1)/3$$

Substituting this reaction function of firm 2 in PR_1 , we can find the optimal location choice of firm 1 by maximizing $PR_1(x_1, x_2(x_1))$, a concave function of x_1 . One obtains the following unique subgame perfect equilibrium :

$$(4) \qquad x_1^* = 2/5 \qquad x_2^* = 4/5$$

$$PR_1^* = 1/5 \qquad PR_2^* = 3/25$$

Proceeding in a similar fashion, we can compute the equilibrium location and price schedule choices if firm 1 has a production cost of $c_1 = c - \alpha$ while firm 2 has a cost of $c_2 = c$ with $\alpha \in [0 , 1/2]$ for simplicity. In the next section, we will assume that firm 1 knows its cost but that firm 2 ignores whether firm 1 has cost c or $c - \alpha$. For $c_1 = c - \alpha$, we have figure 1.



THE BERTRAND EQUILIBRIUM IN DELIVERED PRICES.

FIGURE 1

Following the same argument as before, one can show that

(5) $$p_1 = p_2 = p\,(x\,;\,x_1\,,\,x_2\,,\,\alpha) \;=\; c + \text{Max}\;\{|x - x_1 - \alpha\,|\,,\,|x - x_2|\,]\}$$

is a Bertrand-Nash equilibrium. The new profit functions are as follows :

(6.1) $$PR_1^{\alpha}\,(x_1\,,\,x_2) = \int_0^{(x_1 + x_2 + \alpha)/2} \{[c + x_2 - x] - [c - \alpha + |x - x_1|\,]\}\,dx$$

$$= [(x_2 - x_1)\,(3\,x_1 + x_2)/4] + [(x_1 + x_2)\,\alpha\,/2] + (\alpha^2/2)$$

$$(6.2) \qquad PR_2^{\alpha}(x_1, x_2) = \int_{(x_1 + x_2 + \alpha)/2}^{1} \{[c - \alpha + x - x_1] - [c + |x - x_2|]\}\, dx$$

$$= (x_2 - x_1) - [(x_2 - x_1)(x_1 + 3x_2)/4] + [(x_1 + x_2)\alpha/2] - \alpha - (\alpha^2/4)$$

The stage 1 (long run) Stackelberg solution is obtained by maximizing firm 2's profit, a concave function of x_2 for a given x_1, with respect to x_2 to obtain the location of firm 2 as a function of x_1, namely

$$(7) \qquad x_2^* = x_2(x_1, \alpha) = (2 + x_1 + \alpha)/3$$

which can be substituted in PR_1^{α} to get $PR_1^{\alpha}(x_1, x_2(x_1, \alpha))$, a concave function of x_1. Because of the symmetry of the problem, we can restrict x_1 to the interval $[0, 1/2]$. The constraint $x_1 \leq 1/2$ is only active when $\alpha > 1/8$. So we get

$$(8.1) \qquad x_1^*(\alpha) = \begin{cases} 2/5 + 4\alpha/5 & \text{if } \alpha \leq 1/8 \\ 1/2 & \text{if } 1/8 < \alpha < 1/2 \end{cases}$$

$$(8.2) \qquad PR_1^{*\alpha} = \begin{cases} (1 + 4\alpha + 4\alpha^2)/5 & \text{if } \alpha \leq 1/8 \\ (7 + 32\alpha + 16\alpha^2)/36 & \text{if } 1/8 < \alpha < 1/2 \end{cases}$$

$$(8.3) \qquad x_2^*(\alpha) = \begin{cases} 4/5 + 3\alpha/5 & \text{if } \alpha \leq 1/8 \\ 5/6 + \alpha/3 & \text{if } 1/8 < \alpha < 1/2 \end{cases}$$

$$(8.4) \qquad PR_2^{*\alpha} = \begin{cases} (3 + 18\alpha + 27\alpha^2)/25 & \text{if } \alpha \leq 1/8 \\ (1 + 4\alpha + 4\alpha^2)/12 & \text{if } 1/8 < \alpha < 1/2 \end{cases}$$

Hence, for $\alpha = 1/8$, we have $x_1 = 1/2$ and $x_2 = 7/8$ giving $PR_1^{\alpha} = 0.313$ and $PR_2^{\alpha} = 0.047$; for $\alpha = 1/2$, we have $x_1 = 1/2$ and $x_2 = 1$ giving $PR_1 = 0.75$ and $PR_2 = 0$.

Without entry, firm 1 would be free to locate at the monopoly location and charge monopoly prices, that is to locate at $x_1^M = 1/2$ and charge the reservation price r to each consumer, for a monopoly profit of $\pi_1^M = r - (c - \alpha/2)$ equal to $r - c + (1/4)$ if $\alpha = 1/2$.

3. THE INCOMPLETE INFORMATION CASE : DEFINITIONS AND PRELIMINARIES.

3.1 The Incomplete Information Model

We will assume from now that firm 1 may be of the type c or $c - \alpha$, but that firm 2 is always of type c. Although firm 1 knows its type, firm 2 ignores it but can possibly infer firm 1's type once it observes firm 1's location choice x_1 . Depending on the parameters of the problem, we may find a separating equilibrium in which firm 1's location reveals its type, a pooling equilibrium in which firm 1's location choice is the same in both cases and therefore reveals nothing regarding its production cost, and finally a semi-separating equilibrium in which firm 1's location is chosen in part at random and therefore may or may not reveal to firm 2 the production cost of firm 1.

We can represent the present set-up as a game tree by supposing that a third player, Nature, plays first by choosing firm 1's production cost, either c or $c - \alpha$ with probabilities $\pi^0(c)$ and $\pi^0(c - \alpha)$. In order to simplify the notations we will denote $\pi^0(c - \alpha)$ by π_α^0 and $\pi^0(c)$ by π^0. The value is observed by firm 1 but not by firm 2 which on the other hand observes firm 1's location before deciding on its own location. Location choices x_1 , x_1' by firm 1 represent different arbitrary points on the interval [0 , 1/2]. For each point of the unit interval, firm 2 has an information set consisting of two elements. Once firm 2 has chosen its location, the type of firm 1, that is whether it has production cost $c_1 = c$ or $c_1 = c - \alpha$, is "revealed" to firm 2 ; in the second stage, both firms choose simultaneously their price schedule.

Given that firm 2 knows that firm 1's production cost is either c or $c - \alpha$ given that it knows that firm 1 knows the precise value of its production cost before deciding on x_1 and given that it knows the location chosen by firm 1 before making its own choice x_2 , then firm 2 will rationally try to infer the value of firm 1's production cost from the observation of x_1 . We are looking for a Perfect Bayesian Equilibrium (PBE) of location choices.

Before defining precisely a PBE, we must introduce some notation. A (mixed) strategy of firm 1 is a mapping $\psi_1(.) : C \rightarrow M_1$, where $C = \{c_1\} \in \{c, c - \alpha\}$ is the set of the types of firm 1 and M_1 the set of probability measures defined on [0, 1/2]. We will denote by $\hat{\Pi}_1(.)$ a cumulative distribution for the location x_1 of firm 1 and by $\hat{\Pi}_1(. ; c_1)$

the cumulative selected by $\psi_1(.)$ for c_1. In the case of pure strategies we simply denote by $\hat{x}_1(.) : C \to X_1$, where $X_1 = [0, 1/2]$, the location decision function of firm 1. A mixed strategy of firm 2 is a mapping $\psi_2(.) : X_1 \to M_2$, where M_2 is the set of probability measures defined on $X_2 = [1/2, 1]$, and we will denote by $\hat{\Pi}_2(.)$ a cumulative distribution for the location x_2 of firm 2 and by $\hat{\Pi}_2(. ; x_1)$ the cumulative distribution selected by $\psi_2(.)$ for x_1. In the case of a pure strategy we simply denote by $\hat{x}_2(.) : X_1 \to X_2$ the firm 2 location decision function. Last we need conditional distributions giving for each location in X_1 the posterior probability on C which firm 2 will use in deciding on its location. Let $\pi(. / x_1)$ be the posterior probability function on C given that x_1 has been observed.

A Perfect Bayesian Equilibrium is a couple of strategies $(\psi_1^*(.), \psi_2^*(.))$ and a family of distribution functions $\pi^*(. / x_1)$ such that :

(i) $\forall\, c_1 \in C$

$$\psi_1^*(c_1) \in \underset{\hat{\Pi}_1(.)}{\arg\max} \int_{X_1} EPR_1(x_1 ; \hat{\Pi}_2^*, c_1)\, d\hat{\Pi}_1(x_1)$$

where :

$$EPR_1(x_1 ; \hat{\Pi}_2^*, c_1) = \int_{X_2} PR_1(x_1, x_2, c_1)\, d\hat{\Pi}_2(x_2 ; x_1)$$

(ii) $\forall\, x_1 \in X_1$

$$\psi_2^*(x_1) \in \underset{\hat{\Pi}_2(.)}{\arg\max} \int_{X_2} EPR_2(x_1, x_2)\, d\hat{\Pi}_2(x_2)$$

where :

$$EPR_2(x_1, x_2) = \sum_{c_1 \in C} \pi^*(c_1 / x_1)\, PR_2(x_1, x_2, c_1)$$

(iii) $\forall\, x_1 \in \bigcup_{c_1 \in C} \text{Supp}\, \hat{\Pi}_1^*(. ; c_1)$, $\pi^*(. / x_1)$ is obtained by applying Bayes' rule to the prior distribution $\pi^\circ(.)$ given that x_1 has been observed.

$\forall\, x_1 \notin \bigcup_{c_1 \in C} \text{Supp}\, \hat{\Pi}_1^*(. ; c_1)$, $\pi^*(. / x_1)$ is arbitrary.

For pure strategy equilibria $(\hat{x}_1^*(.), \hat{x}_2^*(.))$, (i), (ii) and (iii) become :

(i') $\forall\, c_1 \in C$

$$\hat{x}_1^*(c_1) \in \arg\max_{x_1} PR_1(x_1, \hat{x}_2^*(x_1), c_1)$$

(ii') $\forall\, x_1 \in X_1$

$$\hat{x}_2^*(x_1) \in \arg\max_{x_2} EPR_2(x_1, x_2)$$

(iii') $\forall\ x_1 \in R\hat{x}_1^*(.)$, $\pi^*(. / x_1)$ is obtained from $\pi^0(.)$ by Bayes' rule, where $R\hat{x}_1^*(.)$ is the range of $\hat{x}_1^*(.)$. $\forall\ x_1 \notin R\hat{x}_1^*(.)$, $\pi^*(. /x_1)$ is arbitrary.

We can now proceed with the analysis and characterization of the different kinds of PBE. We will first consider the separating equilibria in which firm 2 can perfectly infer the value of c_1 from the location choice of firm 1. We will then consider the pooling equilibria in which firm 1 selects the same location for both $c_1 = c$ and $c_1 = c - \alpha$ preventing firm 2 from gaining any information on the value of c_1 from the observed location choice of firm 1. Finally, we will look at the semi-separating equilibria in which firm 1 selects a mixed stategy for either $c_1 = c$ or $c_1 = c - \alpha$; hence the observation of firm 1's location may or may not generate information on c_1 for firm 2.

3.2 Preliminary Results

In order to build all these equilibria we use repeatedly the same kind of arguments. So to save redundancies, it is most useful to introduce now the central concepts and technical apparatus of all the subsequent proofs.

For the sake of notation simplicity we will denote by π_α the probability with which firm 2 is believing that firm 1's cost is $c - \alpha$.

3.2.1) Let us define $\underline{PR}_1(x_1, \pi_\alpha)$ and $\overline{PR}_1(x_1, \pi_\alpha)$ as the profit of, respectively, the low and the high cost firm 1 located at x_1, given that firm 2, observing x_1 and believing that firm 1 is of the low cost type with probability π_α, is optimally located. Formally, let :

$$(9) \qquad \tilde{x}_2(x_1, \pi_\alpha) \in \arg\max_{x_2} \{\pi_\alpha PR_2^a(x_1, x_2) + (1 - \pi_\alpha) PR_2(x_1, x_2)\}$$

Easy calculations show that :

$$(10) \qquad \tilde{x}_2(x_1, \pi_\alpha) = (x_1 + 2 + \pi_\alpha \alpha)/3 .$$

Then

$$(11) \qquad \underline{PR}_1(x_1, \pi_\alpha) = PR_1^a(x_1, \tilde{x}_2(x_1, \pi_\alpha))$$

$$(12) \qquad \overline{PR}_1(x_1, \pi_\alpha) = PR_1(x_1, \tilde{x}_2(x_1, \pi_\alpha))$$

and from (2.1), (6.1), (10), (11) and (12) we get :

$$(13) \qquad \underline{PR}_1(x_1, \pi_\alpha) = (1/36)\{-20(x_1)^2 + 4(4 + 2\pi_\alpha \alpha + 6\alpha)x_1 + (\pi_\alpha \alpha)^2 + 4\pi_\alpha \alpha + 6\pi_\alpha \alpha^2 + 9\alpha^2 + 12\alpha + 4\}$$

$$(14) \qquad \overline{PR}_1(x_1, \pi_\alpha) = (1/36)\{-20(x_1)^2 + 4(4 + 2\pi_\alpha \alpha)x_1 + (\pi_\alpha \alpha)^2 + 4\pi_\alpha \alpha + 4\}$$

Both (13) and (14) are strictly concave functions of x_1.

We will denote by $\underline{x}_1(\pi_\alpha)$ and $\bar{x}_1(\pi_\alpha)$ the firm 1's locations maximizing $\underline{PR}_1(x_1, \pi_\alpha)$ and $\overline{PR}_1(x_1, \pi_\alpha)$ under the constraint $x_1 \leq 1/2$:

$$\underline{x}_1(\pi_\alpha) = \min\{(2 + \pi_\alpha \alpha + 3\alpha)/5, 1/2\} \tag{15}$$

$$\bar{x}_1(\pi_\alpha) = (2 + \pi_\alpha \alpha)/5 \tag{16}$$

Note that $\bar{x}_1(\pi_\alpha)$ and $\underline{x}_1(\pi_\alpha)$, when less than 1/2, are both strictly increasing function of π_α. Furthermore $\bar{x}_1(1) < \underline{x}_1(0)$, except for $\alpha = 1/2$, in which case $\bar{x}_1(1) = \underline{x}_1(\pi_\alpha) = 1/2$ for all π_α. More generally the critical values of α are $\alpha = 1/8$ and $\alpha = 1/6$:

$\alpha \in (0, 1/8) \Rightarrow \underline{x}_1(1) < 1/2 \Rightarrow \forall \pi_\alpha : \underline{x}_1(\pi_\alpha) < 1/2$

$\forall \pi_\alpha : \bar{x}_1(\pi_\alpha) < 1/2$

$\alpha = 1/8 \Rightarrow \underline{x}_1(1) = 1/2 \Rightarrow \forall \pi_\alpha \in [0, 1) : \underline{x}_1(\pi_\alpha) < 1/2$

$\forall \pi_\alpha : \bar{x}_1(\pi_\alpha) < 1/2$

$\alpha \in (1/8, 1/6) \Rightarrow \exists \dot{\pi}_\alpha \in (0, 1) : \underline{x}_1(\dot{\pi}_\alpha) = 1/2$

$\Rightarrow \forall \pi_\alpha \in [\dot{\pi}_\alpha, 1) : \underline{x}_1(\pi_\alpha) = 1/2$

$\forall \pi_\alpha \in [0, \dot{\pi}_\alpha) : \underline{x}_1(\pi_\alpha) < 1/2$

$\forall \pi_\alpha : \bar{x}_1(\pi_\alpha) < 1/2$

$\alpha \in [1/6, 1/2) \Rightarrow \underline{x}_1(0) = 1/2$ and $\bar{x}_1(1) < 1/2$

$\Rightarrow \forall \pi_\alpha : \underline{x}_1(\pi_\alpha) = 1/2$

$\forall \pi_\alpha : \bar{x}_1(\pi_\alpha) < 1/2$

$\alpha = 1/2 \Rightarrow \bar{x}_1(1) = 1/2 \Rightarrow \forall \pi_\alpha : \underline{x}_1(\pi_\alpha) = 1/2$

$\forall \pi_\alpha \in [0, 1) : \bar{x}_1(\pi_\alpha) < 1/2$

3.2.2) Consider now the high cost firm 1. We construct first the interval $[\bar{\bar{x}}_1^{(-)}, \bar{\bar{x}}_1^{(+)}]$ of locations not strictly dominated by $\bar{x}_1(0)$. Let $\bar{\bar{x}}_1^{(-)}$ be defined as either the solution of :

$$\overline{PR}_1 (x_1 , 1) = \overline{PR}_1 (\bar{x}_1 (0), 0) \tag{17}$$

$$0 < x_1 < \bar{x}_1 (0) \tag{18}$$

if such a solution exists, or equal to zero if not. Similarily $\bar{\bar{x}}_1^{(+)}$ is defined as either the solution of both (17) and (19) :

$$\bar{x}_1 (0) < x_1 < 1/2 \tag{19}$$

if such a solution exists, or equal to 1/2. (see figures 2 and 3).

We show in Appendix I that :

$$\bar{\bar{x}}_1^{(-)} = (2 + \alpha - \bar{\Delta}^{1/2})/5 \quad \Rightarrow \quad 0 < \bar{\bar{x}}_1^{(-)} < \bar{x}_1(0) \tag{20}$$

$$\bar{\bar{x}}_1^{(+)} = \begin{cases} (2 + \alpha + \bar{\Delta}^{1/2}) /5 \ , & \text{if } \alpha \in (0, (9/\sqrt{5}) - 4) \Rightarrow \bar{x}_1 (0) < \bar{\bar{x}}_1^{(+)} < 1/2 \\ 1/2 \ , & \text{if } \in [(9/\sqrt{5}) - 4 , 1/2] \end{cases} \tag{21}$$

where $\bar{\Delta} = 9 (\alpha^2 / 4 + \alpha)$.

It must be noticed that any location $x_1' \in [0, \bar{\bar{x}}_1^{(-)})$ is a location strictly dominated by $\bar{x}_1(0)$ for the high cost firm 1 :

$\forall \pi_\alpha , \forall \pi_\alpha$

$$\overline{PR}_1 (x_1' , \pi_\alpha) < \overline{PR}_1 (\bar{x}_1(0), \pi_\alpha) \tag{22}$$

whatever the beliefs of firm 2 observing x_1' , and whathever its beliefs observing $\bar{x}_1(0)$ the high cost firm 1's profits will be higher at $\bar{x}_1(0)$. Likewise, provided that $\bar{\bar{x}}_1^{(+)} < 1/2$, any location $x_1' \in (x_1^{(+)}, 1/2]$ is strictly dominated by $\bar{x}_1(0)$.

For any location $x_1 \in [0,1/2]$, we introduce the probability limit $\bar{\pi}_\alpha(x_1)$ which would induce the high cost firm to prefer the location $\bar{x}_1(0)$ (its highest profit location if rightly identified by the follower) whatever the firm 2's beliefs, rather than the location x_1 and facing a firm 2 with beliefs $\pi_\alpha \leq \bar{\pi}_\alpha(x_1)$. Let us define $\bar{\pi}_\alpha(x_1)$ as either the solution of :

$$\overline{PR}_1(x_1, \pi_\alpha) = \overline{PR}_1 (\bar{x}_1(0),0) \tag{23}$$

if such a solution exists, or equal to 1 if not.

Clearly the function $\bar{\pi}_\alpha(x_1)$ is continuous : first constant and equal to 1 on $[0, \bar{x}_1^{(-)}]$, second strictly decreasing from 1 to 0 on $(\bar{x}_1^{(-)}, \bar{x}_1(0))$, third strictly increasing on $(\bar{x}_1(0), \bar{x}_1^{(+)})$ either from 0 to 1 if $\overline{PR}_1(\bar{x}_1^{(+)}, 1) = \overline{PR}_1(\bar{x}_1(0),0)$ (i.e if $\alpha \in (0, (9/\sqrt{5}) - 4]$) , or from 0 to strictly less than 1 if $\overline{PR}_1(\bar{x}_1^{(+)},1) > \overline{PR}_1(\bar{x}_1(0),0)$ (i.e. if $\alpha > (9/\sqrt{5}) - 4$) and fourth, if $\bar{x}_1^{(+)} < 1/2$ (i.e if $\alpha < (9/\sqrt{5}) - 4$) , constant and equal to 1 on $(\bar{x}_1^{(+)},1/2]$.

Note that :

$$\forall\ x_1 \in [0,1/2] :$$

$$\overline{PR}_1(x_1, \pi_\alpha) \leq \overline{PR}_1(\bar{x}_1(0), 0) \quad \text{iff} \quad \pi_\alpha \leq \bar{\pi}_\alpha(x_1) \tag{24}$$

with a strict inequality if either

$$\pi_\alpha < \bar{\pi}_\alpha(x_1) \text{ or } x_1 \notin [\bar{x}_1^{(-)}, \bar{x}_1^{(+)}].$$

Since $\overline{PR}_1(\bar{x}_1(0),\pi_\alpha)$ is an increasing function of π_α , the high cost firm 1 will prefer to be located at $\bar{x}_1(0)$ facing a firm 2 with any belief rather than to be located at $x_1 \neq \bar{x}_1(0)$ and facing a firm 2 with beliefs $\pi_\alpha \leqslant \bar{\pi}_\alpha(x_1)$.

We repeat the same kind of constructions for any location $\dot{x}_1 \in [\bar{x}_1^{(-)}, \bar{x}_1^{(+)}]$ and beliefs $\dot{\pi}_\alpha \geqslant \bar{\pi}_\alpha(\dot{x}_1)$. We first construct the interval $[\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha), \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)]$ of locations non- strictly dominated by $\dot{x}_1$ if firm 2 beliefs are $\pi_\alpha \leqslant \dot{\pi}_\alpha$ when observing $\dot{x}_1$. Formally let $\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha)$ be defined as either the solution of :

$$\overline{PR}_1(x_1, 1) = \overline{PR}_1(\dot{x}_1, \dot{\pi}_\alpha) \tag{25}$$

$$0 < x_1 < \dot{x}_1 \tag{26}$$

if such solution exists, or equal to 0 if not, and $\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$ as either the solution of (25) and (27) :

$$\dot{x}_1 < x_1 < 1/2 \tag{27}$$

if such a solution exists, or equal to 1/2 if not (see Figures 2 and 3).

Any location $x_1' \in [0, \bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha)]$ is dominated by $\dot{x}_1$ provided that the firm 2's beliefs are such that $\pi_\alpha \geq \dot{\pi}_\alpha$ when observing $\dot{x}_1$, i.e :

$$\forall\ \pi_\alpha',\ \forall\ \pi_\alpha \geq (>)\ \dot{\pi}_\alpha$$

$$\overline{PR}_1(x_1', \pi_\alpha') \leq (<)\ \overline{PR}_1(\dot{x}_1, \pi_\alpha) \tag{28}$$

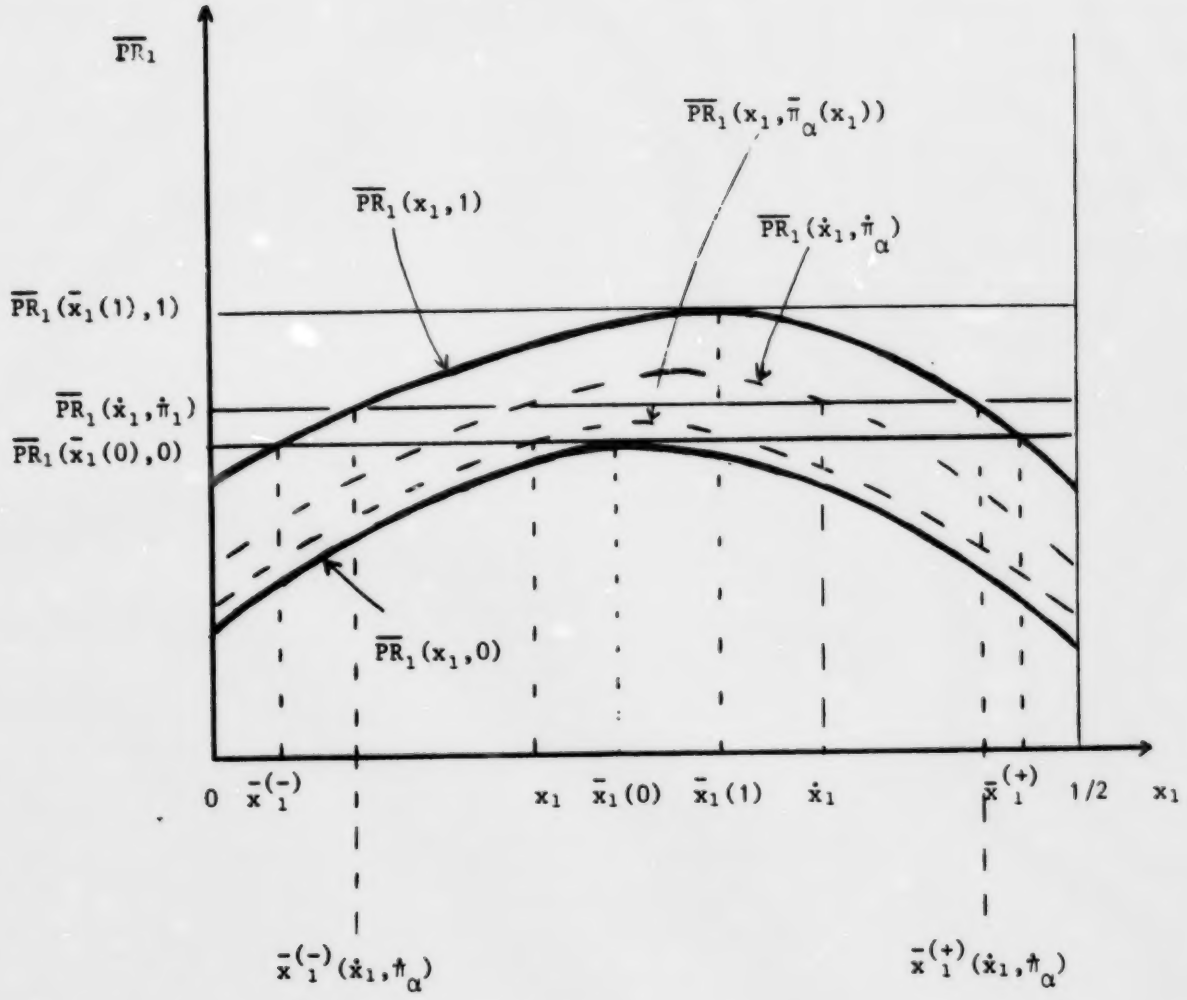
In the same way, if $\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) < 1/2$, any location $x_1' \in (\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_1), 1/2]$ is dominated by $\dot{x}_1$ when the firm 1 is facing a firm 2 with beliefs $\pi_\alpha > \dot{\pi}_\alpha$ when observing $\dot{x}_1$.

For a given pair $(\dot{x}_1, \dot{\pi}_\alpha)$, $\dot{x}_1 \in [\bar{x}_1^{(-)}, \bar{x}_1^{(+)}]$, $\dot{\pi}_\alpha \geq \bar{\pi}_\alpha(\dot{x}_1)$, we define for each location $x_1 \in [0, 1/2]$ the probability limit $\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$ which would imply that firm 1 would be better either at $\dot{x}_1$ and facing a firm 2 with beliefs $\pi_\alpha \geq \dot{\pi}_\alpha$ than at x_1 and facing a firm 2 with beliefs $\pi_\alpha \leq \bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$, or symmetrically better at x_1 and facing a firm 2 with beliefs $\pi_\alpha > \bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$ (if possible) than at $\dot{x}_1$ and facing a firm 2 with beliefs $\pi_\alpha \leq \dot{\pi}_\alpha$.

Formally let $\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$ be defined as either the value of π_α solution of :

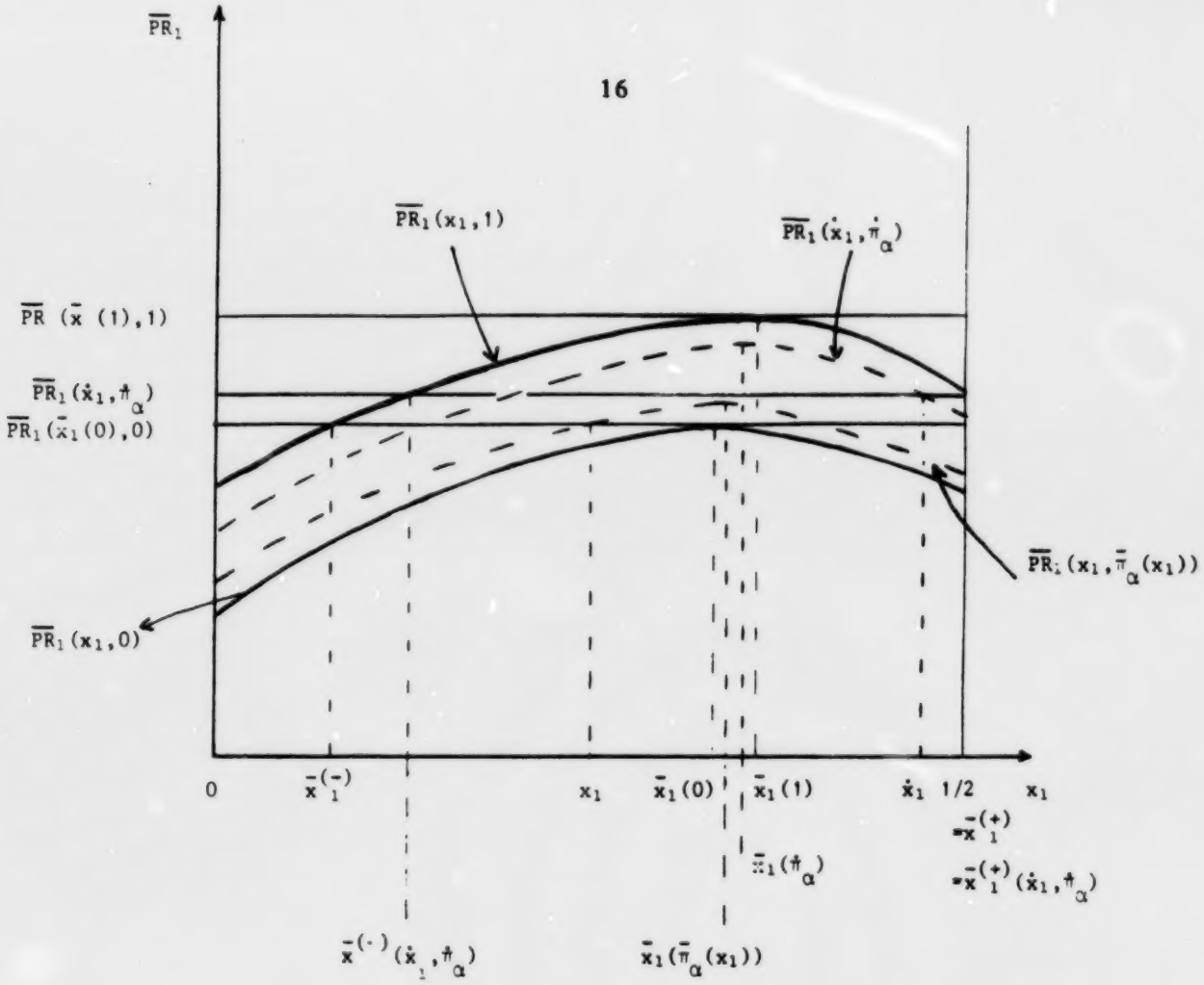
$$\overline{PR}_1(x_1, \pi_\alpha) = \overline{PR}_1(\dot{x}_1, \dot{\pi}_\alpha) \tag{29}$$

if such a solution exists, or equal to 1 if not (see figures 2 and 3).



THE HIGH COST FIRM ONE PROFIT FUNCTION ($\alpha < 9/\sqrt{5} - 4$)

FIGURE 2



THE HIGH COST FIRM ONE PROFIT FUNCTION ($\alpha > 9/\sqrt{5} - 4$)

FIGURE 3

As a function of x_1, $\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$ is first constant and equal to 1 on $[0, \bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha)]$, second strictly decreasing from 1 to some lower bound defined as $\min \{ \pi_\alpha / \exists\ x_1 : \overline{PR}_1(x_1, \pi_\alpha) = \overline{PR}(\dot{x}_1, \dot{\pi}_\alpha) \}$, third strictly increasing from this lower bound either up to 1 if $\overline{PR}_1(\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha), 1) = \overline{PR}_1(\dot{x}_1, \dot{\pi}_\alpha)$, or to less than 1 if $\overline{PR}_1(\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha), 1) > \overline{PR}_1(\dot{x}_1, \pi_\alpha)$(in which case $(\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) = 1/2$), and fourth, if $x_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) < 1/2$,constant and equal to 1.

It must be noticed that :

(30) $\overline{PR}_1(x_1, \pi_\alpha) \lessgtr \overline{PR}_1(\dot{x}_1, \dot{\pi}_\alpha)$ iff $\pi_\alpha \lessgtr \bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$

with the strict inequality if either $\pi_\alpha < \bar{\pi}^\alpha(x_1, \dot{x}, \dot{\pi}_\alpha)$

or $x_1 \notin [\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha), \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)]$.

So that the high cost firm 1 would prefer to be located at $\dot{x}_1$ and facing a firm 2 with beliefs $\pi_\alpha \geqslant \dot{\pi}_\alpha$ rather than at x_1 and facing a firm 2 with beliefs $\pi_\alpha \leqslant \bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$.

3.2.3) We can define for the low cost firm 1 the same kind of location intervals and probability limits.

We first define the interval $[\, \underline{x}_1^{(-)}, \underline{x}_1^{(+)}\,]$ of locations non strictly dominated by $\underline{x}_1(0)$ for the low cost firm 1. We show in Appendix I that :

$$(31)\qquad \underline{x}_1^{(-)} \begin{cases} (2(1+2\alpha) - \Delta'^{1/2})/5 & \text{if } \alpha \in (0\ ,\ 1/6] \\ (4(1+2\alpha) - \Delta'^{1/2})/10 & \text{if } \alpha \in [1/6,\ 1/2]. \end{cases}$$

$$\Rightarrow 0 < \underline{x}_1^{(-)} < \underline{x}_1(0)$$

where $\Delta' = 99\alpha^2 + 24\alpha + 1$

$$(32)\qquad \underline{x}_1^{(+)} = \begin{cases} (2(1+2\alpha) + \Delta'^{1/2})/5 & \text{if } \alpha \in (0\ ,\ 26 - (26^2 - 1)^{1/2}) \\ & \Rightarrow \underline{x}_1(0) < \underline{x}_1^{(+)} < 1/2 \\ 1/2 & \text{if } \alpha \in [\ 26 - (26^2 - 1)^{1/2}\ ,\ 1/2] \end{cases}$$

where $\Delta = 63\alpha^2 / 4 + 9\alpha$.

From (20), (21), (31) and (32) it follows that :

$$(33)\qquad \bar{x}_1^{(-)} < \underline{x}_1^{(-)} < \bar{x}_1^{(+)} \leq \underline{x}_1^{(+)}$$

with a strict inequality iff $\bar{x}_1^{(+)} < 1/2$.

The probability limit $\underline{\Pi}_\alpha(x_1)$, defined in a similar way as $\bar{\pi}_\alpha(x_1)$, is the benchmark of beliefs of firm 2 observing x_1 , under which it would be better for the low cost firm 1 to be located at $\underline{x}_1(0)$ rather than at x_1 :

$$\forall\ x_1 \in [\,0,\ 1/2\,]$$

$$(34)\qquad \underline{PR}_1(x_1\ ,\ \pi_\alpha) \leqslant \underline{PR}_1(\underline{x}_1(0),\ 0) \text{ iff } \pi_\alpha \leqslant \underline{\Pi}_\alpha(x_1)$$

with the strict inequality if either $\pi_\alpha < \underline{\Pi}_\alpha(x_1)$ or $x_1 \notin [\, \underline{x}_1^{(-)}\ ,\ \underline{x}_1^{(+)}\,]$.

The $\underline{\Pi}_\alpha(x_1)$ function is continuous : first equal to 1 on $[0\ ,\ \underline{x}_1^{(-)}]$, second stictly decreasing from 1 to 0 on $[\underline{x}_1^{(-)}\ ,\ \underline{x}_1(0)]$, third if $\underline{x}_1(0) < 1/2$ (i.e. if $\alpha < 1/6$) strictly increasing on $(\underline{x}_1(0)\ ,\ \underline{x}_1^{(+)})$ either from 0 to 1 if $\underline{PR}_1(\underline{x}_1^{(+)}, 1) = \underline{PR}_1(\underline{x}_1(0), 0)$, or from 0 to strictly less than 1 if $\underline{PR}_1(\underline{x}_1^{(+)}, 1) > \underline{PR}_1(\underline{x}_1(0), 0)$, and fourth , if $\underline{x}_1^{(+)} < 1/2$, constant and equal to 1 on $(\underline{x}_1^{(+)}, 1]$.

For any location $\dot{x}_1 \in [\underline{x}_1^{(-)}\ ,\underline{x}_1^{(+)}]$ and $\dot{\pi}_\alpha \geq \underline{\Pi}_\alpha(x_1)$ we define also for the low cost firm the interval $[\underline{x}_1^{(-)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha)\ ,\ \underline{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha)]$ of locations non strictly dominated by $\dot{x}_1$, when firm 2's beliefs observing $\dot{x}_1$ are $\pi_\alpha \leq \dot{\pi}_\alpha$. It is demonstrated in Appendix II that :

$$\forall\ \dot{x}_1 \in [\bar{x}_1^{(-)}\ ,\bar{x}_1^{(+)}] \cap [\underline{x}_1^{(-)}\ ,\underline{x}_1^{(+)}] = [\underline{x}_1^{(-)}\ ,\bar{x}_1^{(+)}]$$

$$\forall \pi_\alpha \geq \max\{\bar{\pi}_\alpha(\dot{x}_1)\ ,\underline{\Pi}_\alpha(\dot{x}_1)\}$$

$$(35) \qquad \bar{x}_1^{(-)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha) \leqslant \underline{x}_1^{(-)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha) \leqslant \bar{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha) \leqslant \underline{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha)$$

with :

- the first inequality as an equality if $\dot{x}_1 < \bar{x}_1(1)$ and $\dot{\pi}_\alpha = 1$
- the first and second inequalities as equalities if $\dot{x}_1 = \bar{x}_1(1)$ and $\dot{\pi}_\alpha = 1$
- the second inequality as an equality if $\dot{x}_1 \in (\bar{x}_1(1),\ \underline{x}_1(1))$ and $\dot{\pi}_\alpha = 1$
- the second and third inequalities as equalities if $\dot{x}_1 = \underline{x}_1(1)$ and $\dot{\pi}_\alpha = 1$
- the third inequality as an equality if either $\dot{x}_1 > \underline{x}_1(1)$ and $\dot{\pi}_\alpha = 1$ or $\bar{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_\alpha) = 1/2$.

Finally for a given pair $(\dot{x}_1\ ,\ \dot{\pi}_\alpha)$, $\dot{x}_1 \in [\underline{x}_1^{(-)}\ ,\ \underline{x}_1^{(+)}]$ and $\dot{\pi}_\alpha \geqslant \underline{\Pi}_\alpha(x_1)$ we define for each location $x_1 \in [0\ ,\ 1/2]$ the probability limit $\underline{\pi}^\alpha(x_1\ ,\ \dot{x}\ ,\ \dot{\pi}_\alpha)$ in a similar way to $\bar{\pi}^\alpha(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)$ so that :

$$\forall\ x_1 \in [0\ ,\ 1/2]$$

$$(36) \qquad \underline{PR}_1(x_1\ ,\pi_\alpha) \leqslant \underline{PR}_1(\dot{x}_1\ ,\ \dot{\pi}_\alpha) \text{ iff } \pi_\alpha \leqslant \underline{\pi}^\alpha(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)$$

with the strict inequality if either $\pi_\alpha < \underline{\pi}^\alpha(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)$

or $x_1 \notin [\underline{x}_1^{(-)}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)\ ,\ \underline{x}_1^{(+)}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)]$

and the low cost firm 1 would be better at $\dot{x}_1$ and facing a firm 2 with beliefs $\pi_\alpha \geqslant \dot{\pi}_\alpha$ rather than at x_1 and facing a firm 2 with beliefs $\pi_\alpha \leqslant \underline{\pi}^\alpha(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_\alpha)$.

Note that for $\alpha \in [1/6\ ,\ 1/2]$:

$$\underline{\pi}^\alpha(x_1) = \underline{\pi}^\alpha(x_1,\ 1/2,\ 0) \text{ and } \underline{x}_1^{(-)} = \underline{x}_1^{(-)}(1/2,\ 0).$$

3.2.4) For a given pair of location $\dot{x}_1$ and beliefs $\dot{\pi}_\alpha$ with $\dot{x}_1 \in [\underline{x}^{(-)}_1 , \bar{x}^{(+)}_1]$ and $\dot{\pi}_\alpha \geqslant \max \{\bar{\pi}_\alpha(\dot{x}_1), \underline{\pi}_\alpha(\dot{x}_1)\}$, we have, as shown in Appendix II, the following relations between $\bar{\pi}_\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$ and $\underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$:

(*) by definition :

(37) $$\bar{\pi}^\alpha(\dot{x}_1, \dot{x}_1, \dot{\pi}_\alpha) = \underline{\pi}^\alpha(\dot{x}_1, \dot{x}_1, \dot{\pi}_\alpha)$$

(38) $$\forall\ x_1 \lessdot \min \{\bar{x}^{(-)}_1(\dot{x}_1, \dot{\pi}_\alpha), \underline{x}^{(-)}_1(\dot{x}_1, \dot{\pi}_\alpha)\}$$

$$\forall\ x_1 \gtrdot \max \{\bar{x}^{(+)}_1(\dot{x}_1, \dot{\pi}_\alpha), \underline{x}^{(+)}_1(\dot{x}_1, \dot{\pi}_\alpha)\}$$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) = \underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) = 1$$

(*) for $\dot{\pi}_\alpha = 1$ and $\dot{x}_1 \lessdot \bar{x}_1(1)$

(39) $$\dot{x}_1 = \bar{x}^{(-)}_1(\dot{x}_1, 1) = \underline{x}^{(-)}_1(\dot{x}_1, 1) \lessdot \bar{x}^{(+)}_1(\dot{x}_1, 1) \lessdot \underline{x}^{(+)}_1(\dot{x}_1, 1)$$

with the third inequality as an equality if $\dot{x}_1 = \bar{x}_1(1)$ and a fourth strict inequality if $\bar{x}^{(+)}_1(\dot{x}_1, 1) < 1/2$

(40) $$\forall\ x_1 \in (\dot{x}_1, \underline{x}^{(+)}_1(\dot{x}_1, 1))$$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, 1) = \underline{\pi}^\alpha(x_1, \dot{x}_1, 1)$$

(*) for $\dot{\pi}_\alpha = 1$ and $\dot{x}_1 \in (\bar{x}_1(1), \underline{x}_1(1))$

(41) $$\bar{x}^{(-)}_1(\dot{x}_1, 1) < \bar{x}^{(+)}_1(\dot{x}_1, 1) = \dot{x}_1 = \underline{x}^{(-)}_1(\dot{x}_1, 1) < \underline{x}^{(+)}_1(\dot{x}_1, 1)$$

(42) $$\forall\ x_1 \in (\bar{x}^{(-)}_1(\dot{x}_1, 1), \dot{x}_1)$$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, 1) < \underline{\pi}^\alpha(x_1, \dot{x}_1, 1) = 1$$

(43) $$\forall\ x_1 \in (\dot{x}_1, \underline{x}^{(+)}_1(\dot{x}_1, 1))$$

$$1 = \bar{\pi}^\alpha(x_1, \dot{x}_1, 1) > \underline{\pi}^\alpha(x_1, \dot{x}_1, 1)$$

(44) (*) for $\dot{\pi}_\alpha > 1$ and $\dot{x}_1 \gtrdot \underline{x}_1(1)$:

$$x^{(-)}_1(\dot{x}_1, \dot{\pi}_\alpha) < \underline{x}^{(-)}_1(\dot{x}_1, \dot{\pi}_\alpha) \lessdot \dot{x}_1 = \bar{x}^{(+)}_1(\dot{x}_1, \dot{\pi}_\alpha) = \underline{x}^{(+)}_1(\dot{x}_1, \dot{\pi}_\alpha)$$

with the second inequality as an equality iff $\dot{x}_1 = \underline{x}_1(1)$

(45) $$\forall\ x_1 \in (\bar{x}^{(-)}_1(\dot{x}_1, \dot{\pi}_\alpha), \dot{x}_1)$$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) < \underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$$

(46) (*) for $\dot{\pi}_\alpha < 1$ and $\dot{x}_1 \leqslant \underline{x}_1(\dot{\pi}_\alpha)$:

$$\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha) < \underline{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha) < \dot{x}_1 \leqslant \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) \leqslant \underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$$

with the third inequality as an equality iff $\dot{x}_1 = 1/2$

and a fourth strict inequality iff $\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) < 1/2$

(47) $\forall\, x_1 \in (\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha), \dot{x}_1)$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) < \underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$$

(48) $\forall\, x_1 \in (\dot{x}_1, \underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha))$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) > \underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$$

(49) (*) for $\dot{\pi}_\alpha < 1$ and $\dot{x}_1 > \underline{x}_1(\dot{\pi}_\alpha)$:

$$\underline{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha) < \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) < \dot{x}_1 \leqslant \underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) \leqslant \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$$

with the third inequality as an equality iff $\dot{x}_1 = 1/2$

and a fourth strict inequality iff $\underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) < 1/2$

(50) $\forall\, x_1 \in (\underline{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha), \dot{x}_1)$

$$\bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) < \underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$$

(51) $\forall\, x_1 \in (\dot{x}_1, \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha))$

$$\underline{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha) < \bar{\pi}^\alpha(x_1, \dot{x}_1, \dot{\pi}_\alpha)$$

4. SEPARATING EQUILIBRIA (SE)

We must distinguish according to the values of α. For α sufficiently low there exists a continuum of pure strategy equilibria the measure of which is a continuous function of α, first increasing from 0 at $\alpha = 0$ and second decreasing and going back to 0 at $\alpha = (9/\sqrt{5}) - 4$. For α higher than this limit there exists no pure strategy separating equilibrium. When there exists a continuum of equilibria the only one which survives Kreps' intuitive criterion (KIC) is the most beneficial to firm 1. It is also the most favorable to firm 2 . Futhermore this equilibrium is also surviving the Cho and Kreps D1 criterion (Cho and Kreps (1988)). Since by the Cho and Sobel theorem (see section 7) there exists only one equilibrium satisfying D1, we need not consider mixed strategy separating equilibria in this region.

4.1 The Set of Separating Equilibria.

Let us begin with the high cost firm 1. In a SE we must have $\hat{x}_1^*(c) = \bar{x}_1(0)$. At this kind of equilibrium the high cost firm is perfectly identified. So suppose that $\hat{x}_1^*(c) = x_1 \neq \bar{x}_1(0)$. Then because $\overline{PR}_1(x_1, 0) < \overline{PR}_1(\bar{x}_1(0), 0) \leqslant \overline{PR}_1(\bar{x}_1(0), \pi_\alpha)$ whatever π_α, for any posterior beliefs of firm 2 the high cost firm would deviate from x_1 to $\bar{x}_1(0)$.

The constraints on the posterior beliefs sustaining $\hat{x}_1^*(c) = \bar{x}_1(0)$ are given by $\pi^*(c - \alpha / x_1) \leqslant \bar{\pi}_\alpha(x_1)$. Hence the posterior beliefs must be strictly less than 1 on $(\bar{x}_1^{(-)}, \bar{x}_1^{(+)})$; and on $(\bar{x}_1^{(-)}, 1/2]$ if furthermore $\overline{PR}_1(\bar{x}_1^{(+)}, 1) > \overline{PR}_1(\bar{x}_1(0), 0)$, i.e if $\alpha > (9/\sqrt{5}) - 4$, since in this case $\bar{\pi}_\alpha(\bar{x}_1^{(+)}) < 1$ and $\bar{x}_1^{(+)} = 1/2$.

As far as the low cost firm is concerned its separating equilibrium location cannot be in the interval $(\bar{x}_1^{(-)}, \bar{x}_1^{(+)})$; nor in the interval $(\bar{x}_1^{(-)}, 1/2]$ if $\alpha > (9/\sqrt{5}) - 4$. $\hat{x}_1^*(c - \alpha)$ in these intervals would imply, since firm 1 is identified, $\pi^*(c - \alpha / \hat{x}_1^*(c - \alpha)) = 1$ violating the constraints on the posterior beliefs imposed by the equilibrium location of the high cost firm 1. Note also that $\hat{x}_1^*(c - \alpha)$ cannot be in $[0, \bar{x}_1^{(-)}]$ for the following reason. Since for all $\alpha \in (0, 1/2]$ we have $\bar{x}_1^{(-)} < \underline{x}_1^{(-)}$, then for all $\alpha \in (0, 1/2]$ and any $\dot{x}_1 \in [0, \bar{x}_1^{(-)}]$: $\underline{PR}_1(\dot{x}_1, 1) < \underline{PR}_1(\underline{x}_1(0), 0)$; but $\underline{PR}_1(\dot{x}_1, \pi_\alpha)$ is strictly increasing in π_α , hence $\underline{PR}_1(\dot{x}_1, 1) < \underline{PR}_1(\underline{x}_1(0), \pi_\alpha)$, $\forall \pi_\alpha$, and the low cost firm 1 would deviate from $\dot{x}_1$ to $\underline{x}_1(0)$.

We first conclude that for $\alpha > (9/\sqrt{5}) - 4$ where $\bar{x}^{(+)}_1 = 1/2$ there cannot exist any separating equilibrium and we suppose from now that $\alpha \leq (9/\sqrt{5}) - 4$.

If $\underline{x}^{(+)}_1 < 1/2$, $\hat{x}^*_1(c - \alpha)$ cannot be in $(\underline{x}^{(+)}_1, 1/2]$. For any $\hat{x}^*_1(c - \alpha)$ in this interval $\underline{PR}_1(\hat{x}^*_1(c - \alpha), 1) < \underline{PR}_1(\underline{x}_1(0), 0) \leq \underline{PR}_1(\underline{x}_1(0), \pi_\alpha), \forall \pi_\alpha$ and the low cost firm 1 would deviate to $\underline{x}_1(0)$. Therefore $\hat{x}^*_1(c - \alpha)$ must be in the interval $[\bar{x}^{(+)}_1, \underline{x}^{(+)}_1]$, reduced to the only point 1/2 in the limit case $\alpha = (9/\sqrt{5}) - 4$, and any posterior beliefs of firm 2 such that $\pi^*(c - \alpha / x_1) \leq \Pi^\alpha(x_1, \hat{x}^*_1(c - \alpha), 1)$, for $x_1 \neq \hat{x}^*_1(c - \alpha)$, would prevent any deviation of the low cost firm 1.

We gather our results in the next proposition.

Proposition 1 :

i) if $\alpha \in (0, (9/\sqrt{5}) - 4)$ there exists a continuum of SE :

$$\hat{x}^*_1(c) = \bar{x}_1(0) \qquad \hat{x}^*_1(c - \alpha) \in [\bar{x}^{(+)}_1, \underline{x}^{(+)}_1]$$

each one supported by any posterior beliefs function $\pi^*(\ / x_1)$ such that :

$$x_1 = \hat{x}^*_1(c_1),\ c_1 \in C, \Rightarrow \pi^*(c_1 / x_1) = 1$$

$$x_1 \neq \hat{x}^*_1(c_1),\ c_1 \in C, \Rightarrow$$

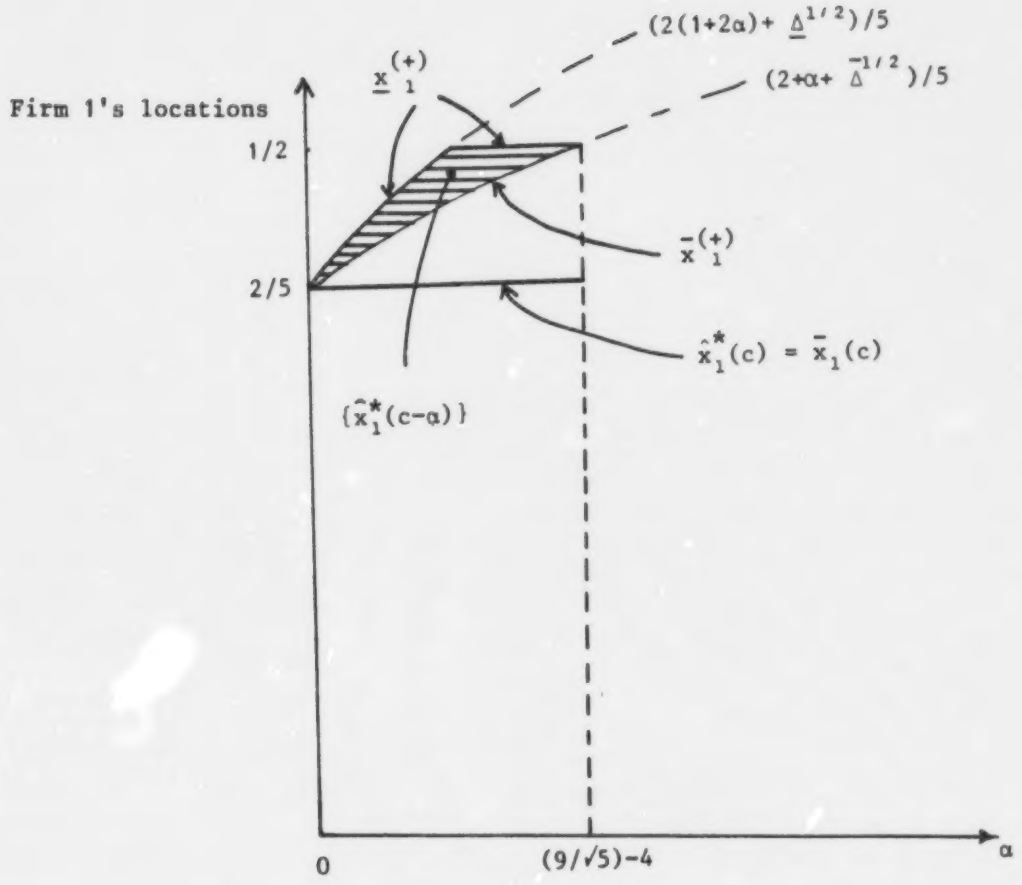
$$\pi^*(c - \alpha / x_1) \leq \min \{ \bar{\pi}_\alpha(x_1), \Pi^\alpha(x_1, \hat{x}^*_1(c - \alpha), 1)\}$$

ii) if $\alpha = (9/\sqrt{5}) - 4$ there exists only one SE :

$$\hat{x}^*_1(c) = \bar{x}_1(0) \qquad \hat{x}^*_1(c - \alpha) = \bar{x}^{(+)}_1 = 1/2$$

supported by the same kind of posterior beliefs.

The graph of the equilibrium correspondence, as a function of α, is illustrated in figure 4.



THE SET OF SEPARATING EQUILIBRIA

FIGURE 4

4.2 The Set of Separating Equilibria Surviving Kreps' Intuitive Criterion.

We demonstrate now that the only equilibrium which survives KIC is $\{\hat{x}_1^*(c) = \bar{x}_1(0),\ \hat{x}_1^*(c-\alpha) = \bar{x}_1^{(+)}\}$. Suppose that $\hat{x}_1^*(c-\alpha) \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}]$ and let an out of equilibrium location $\dot{x}_1 \in (\bar{x}_1^{(+)}, \hat{x}_1^*(c-\alpha))$. Since $\dot{x}_1 > \bar{x}_1^{(+)}$ we have first :

$\forall\ \pi_\alpha$:

$$\overline{PR}_1(\dot{x}_1, \pi_\alpha) < \overline{PR}_1(\bar{x}_1(0), 0) = \overline{PR}_1(\hat{x}_1^*(c), 0)$$

hence whatever the posterior beliefs of firm 2 the high cost firm 1 would not take advantage of this deviation ; second, since $\underline{PR}_1(\dot{x}_1, 1)$ is decreasing on $[\bar{x}_1^{(+)}, \underline{x}_1^{(+)}]$

(it is demonstrated in Appendix 1 that $\underline{x}_1(1) < \bar{x}^{(+1)}$), we get :

$$\underline{PR}_1(\dot{x}_1, 1) > \underline{PR}_1(\hat{x}_1^*(c - \alpha), 1)$$

and the low cost firm, perfectly identified, would profit from the deviation.

The only equilibrium selected by KIC is thus the equilibrium which maximizes firm 1's profits (among the separating equilibria). Since firm 2's equilibrium profits are decreasing with x_1 , its profits are also maximized. Hence with regards to the profits of the firms this equilibrium Pareto-dominates all the separating equilibria.

The location $\hat{x}_1^*(c - \alpha) = \bar{x}^{(+1)}$ is the location which minimizes the cost of separation for firm 1 of type $c - \alpha$. It represents the smallest deviation from its (ex ante) full information equilibrium location $\underline{x}_1$ (1), which allows firm 2 to identify it with certainty, that is to generate an (ex post) full information equilibrium.

4.3 The Set of Separating Equilibria Surviving D1

Finally look at the working of the Cho and Kreps D1 criterion in this model. For any given location x_1 and any firm 2's posterior beliefs there exists only one firm 2's best response in pure strategy and both $\overline{PR}_1(x_1 \ \pi_\alpha)$ and $\underline{PR}_1(x_1, \pi_\alpha)$ are increasing functions of π_α. So the sets D_t and D_t^0 in the Cho and Kreps notation, where t is the firm 1's type, can be identified to the following ranges of posterior beliefs, denoting D_t and D_t^0 by $\bar{D}$ and $\bar{D}^0$ respectively for the high cost firm 1 and by $\underline{D}$ and $\underline{D}^0$ for the low cost one :

$$\bar{D}(x_1) = \{ \pi_\alpha / \overline{PR}_1(x_1, \pi_\alpha) > \overline{PR}_1^* \}$$

$$\bar{D}^0(x_1) = \{ \pi_\alpha / \overline{PR}_1(x_1, \pi_\alpha) = \overline{PR}_1^* \}$$

where $\overline{PR}_1^*$ is the high cost firm 1 equilibrium profit and, similarily :

$$\underline{D}_1(x_1) = \{ \pi_\alpha / \underline{PR}_1(x_1, \pi_\alpha) > \underline{PR}_1^* \}$$

$$\underline{D}_1^0(x_1) = \{ \pi_\alpha / \underline{PR}_1(x_1, \pi_\alpha) = \underline{PR}_1^* \}$$

where $\underline{PR}_1^*$, is the low cost firm 1 equilibrium profit.

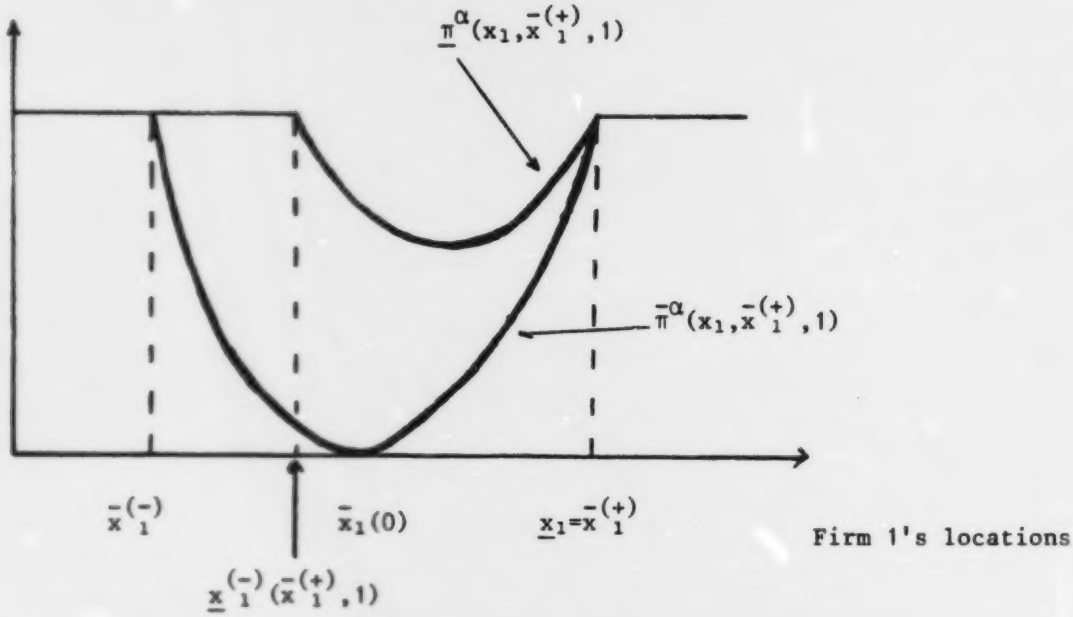
For the equilibrium $\{\hat{x}_1^*(c) = \bar{x}_1(0),\ \hat{x}_1^*(c-\alpha) = \bar{x}_1^{(+)}\}$ which survives KIC we have for any $x_1 \in (\bar{x}_1^{(-)}, \bar{x}_1^{(+)})$:

$$\bar{\pi}_\alpha(x_1) = \bar{\pi}^\alpha(x_1, \hat{x}_1^*(c), 1) < \underline{\pi}^\alpha(x_1, \hat{x}_1^*(c-\alpha), 1)$$

as shown in figure 5, so that :

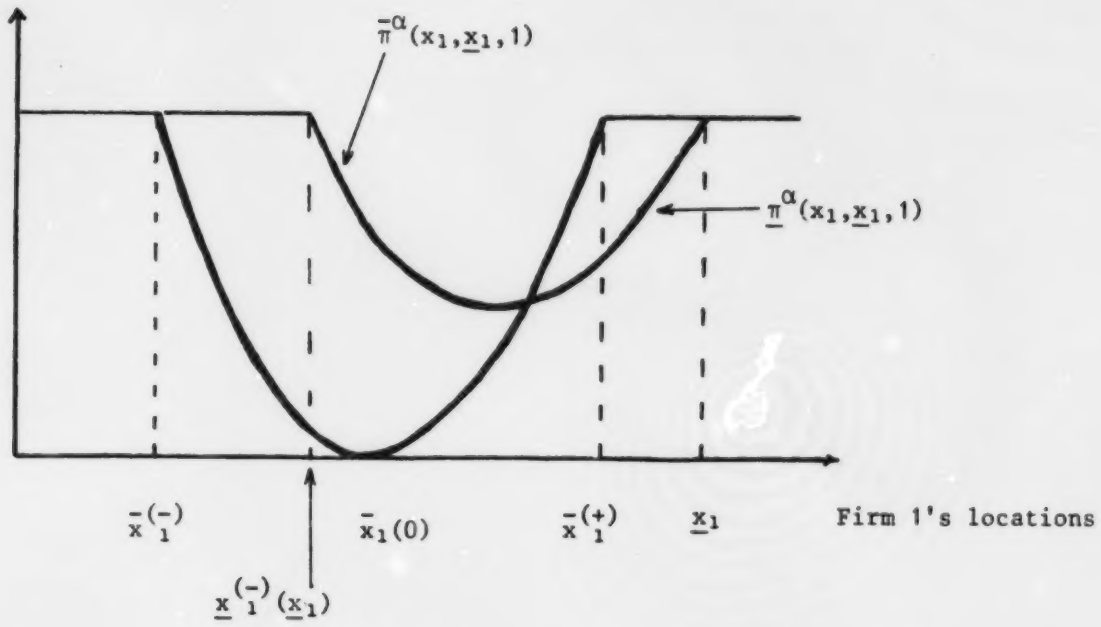
$$\underline{D}(x_1) \cup \underline{D}^0(x_1) \subset \bar{D}(x_1)$$

(with $\underline{D}_1(x_1) = \phi$ and $\underline{D}^0(x_1) = \phi$ if $x_1 < \underline{x}_1^{(-)}(\bar{x}_1^{(+)})$) and the firm 2 should conclude it is facing a high cost firm 1. Hence neither the high nor the low cost firm 1 wants to deviate from its equilibrium location to any $x_1 \in (\bar{x}_1^{(-)}, \bar{x}_1^{(+)})$. Since they do not want to deviate to any other out of equilibrium location, this equilibrium survives D1.



THE ONLY SEPARATING EQUILIBRIUM SURVIVING D1.

FIGURE 5a



ONE OF THE SEPARATING EQUILIBRIA WHICH DOES NOT SURVIVE D1.

FIGURE 5b

Proposition 2

For $\alpha \in (0, 9\sqrt{5} - 4]$ the only PBE surviving the D1 criterion is the separating equilibrium :

$$x_1^*(c) = \bar{x}_1(0), \quad \hat{x}_1^*(c - \alpha) = \bar{x}_1^{(+)}$$

5. POOLING EQUILIBRIA (PE)

In the case of pooling equilibria we must distinguish both according to the values of α and the values of $\pi^{\circ}{}_{\alpha}$, the prior beliefs. For any α there is a whole range of pooling equilibria, the measure of which is increasing with α (see figure 8), generically sustained by a continuum of prior beliefs and subsequent Bayesian belief functions. It happens that the same location is a robust equilibrium against KIC for some priors but not for others. We will say that a pooling location x_1^* generically survives KIC if the measure of the set of priors for which it survives is positive. (Note that this genericity problem is specific to PE and semi-pooling equilibria since the priors do not play any role in the construction of SE). When there exists SE, i.e for $\alpha < (9/\sqrt{5}) - 4$, no PE generically survive. But, when they are generically surviving, not all the priors for which the location is a PE are robust. The measure of the set of eliminated PE tends to zero when α increases toward some bound $\alpha^k < 1/2$. For α higher than α^k all the PE generically survive. Then there appears some subset of PE which survive for all the priors sustaining them. When α tends to 1/2 that set becomes the whole set of PE (see figure 9). However the only PE surviving D1 is the one in which the two types of firm 1 locate at the center of the market provided that the prior probability that firm 1 is of the low cost type is higher than some bound. Again by the Cho and Sobel uniqueness theorem, we need not search for other types of equilibria in this region.

5.1. The Set of Pooling Equilibria

In a PE, by definition $\hat{x}_1^*(c - \alpha) = \hat{x}_1^*(c)$; the same equilibrium location for the two types of firm 1 will be denoted by x_1^* ; and firm 2's posterior beliefs at x_1^* are the prior ones.

Note first that x_1^* can be neither in the interval $(0, \bar{x}_1^{(-)})$ nor in the interval $(\bar{x}_1^{(+)}, 1/2)$ if $\bar{x}_1^{(+)} < 1/2$. Suppose it is the case, then :

$$\overline{PR}_1(x_1^*, \pi'_{\alpha}) < \overline{PR}_1(\bar{x}_1(0), \pi_{\alpha}), \quad \forall \pi'_{\alpha} \ \forall \pi_{\alpha}.$$

Hence whatever the prior beliefs $\pi^{\circ}{}_{\alpha}$ of firm 2 :

$$\overline{PR}_1(x_1^*, \pi^{\circ}{}_{\alpha}) < \overline{PR}_1(\bar{x}_1(0), \pi_{\alpha}), \quad \forall \pi_{\alpha}$$

so whatever the posterior beliefs of firm 2 observing the deviation $\bar{x}_1(0)$ the high cost firm 1's profits would be higher at $\bar{x}_1(0)$ than at x_1^*. A similar argument would

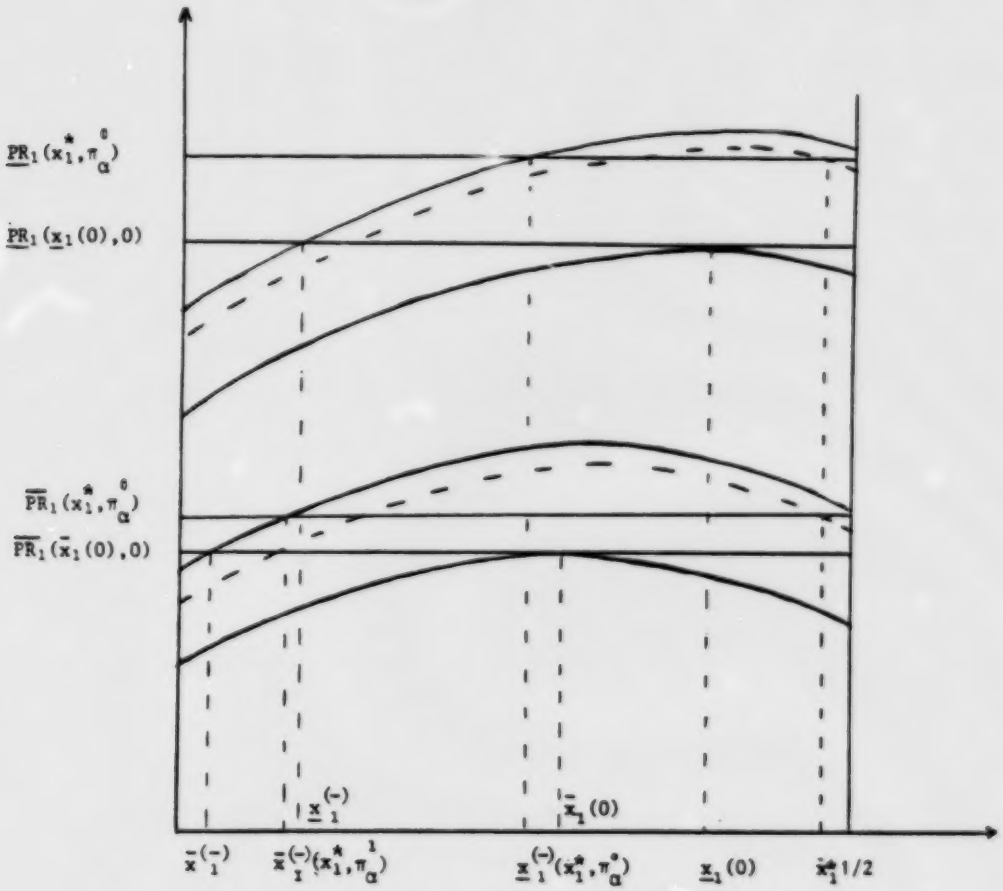
show that x_1^* can be neither in the interval $[0, \underline{x}_1^{(-)})$ nor in the interval $(\underline{x}_1^{(+)}, 1/2]$ if $\underline{x}_1^{(+)} < 1/2$.

We now show that for any $x_1^* \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}] = [\bar{x}_1^{(-)}, \bar{x}_1^{(+)}] \cap [\underline{x}_1^{(-)}, \underline{x}_1^{(+)}]$ there exists some firm 2's prior beliefs π^0_α and some Bayesian posterior beliefs function $\pi^*(\,|x_1)$ which support x_1^* as a pooling equilibrium.

For any $x_1^* \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}]$ take any prior π^0_α such that :

$$\pi_\alpha^0 > \max\{\bar{\pi}^\alpha(x_1^*), \underline{\pi}_\alpha(x_1^*)\}. \tag{52}$$

Given the assumed location x_1^* such a π_α^0 exists. From this inequality it follows that the functions $\bar{\pi}^\alpha(x_1, x_1^*, \pi_\alpha^0)$ and $\underline{\pi}^\alpha(x_1, x_1^*, \pi_\alpha^0)$ are well defined. (see figure 6).



THE POOLING EQUILIBRIUM AT (x_1^*, π_α^0)

FIGURE 6

Let :

- for $x_1 = x_1^*$

$$\pi^*(c - \alpha / x_1^*) = \pi_\alpha^{\circ}$$

- for $x_1 \neq x_1^*$

$$\pi^*(c - \alpha / x_1) \leq \min \{ \bar{\pi}^*(x_1, x_1^*, \pi_\alpha^{\circ}), \underline{\pi}^*(x_1, x_1^*, \pi_\alpha^{\circ}) \}$$

then :

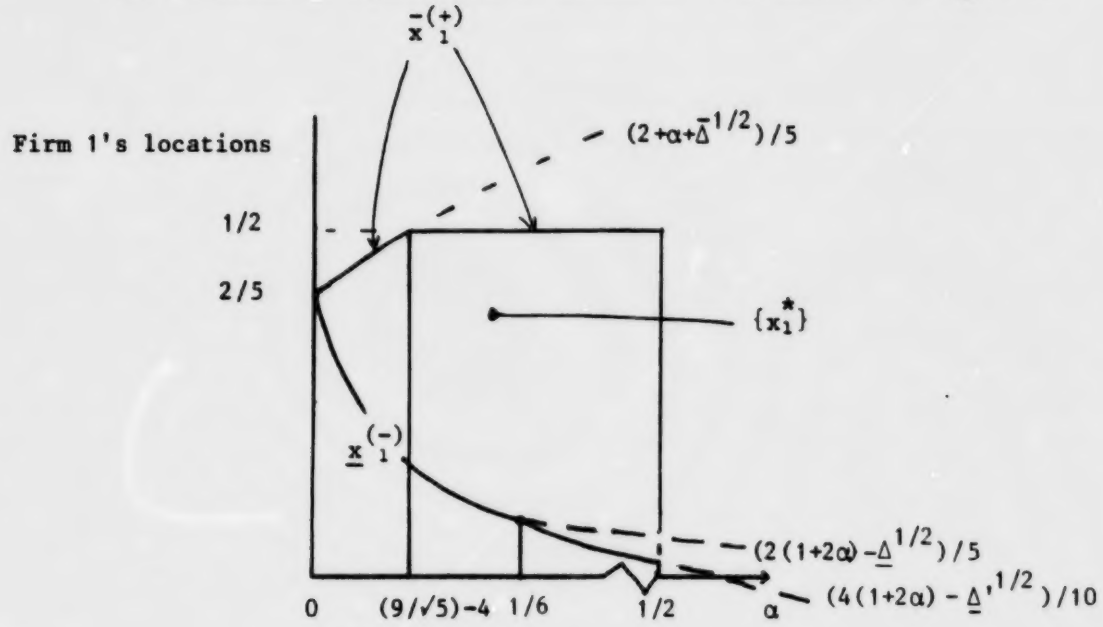
$$\overline{PR}_1(x_1, \pi^*(c - \alpha / x_1)) \leq \overline{PR}_1(x_1, \bar{\pi}^\alpha(x_1, x_1^*, \pi_\alpha^{\circ})) \leq \overline{PR}_1(x_1^*, \pi_\alpha^{\circ})$$

and

$$\underline{PR}_1(x_1, \pi^*(c - \alpha / x_1)) \leq \underline{PR}_1(x_1, \underline{\pi}^\alpha(x_1, x_1^*, \pi_\alpha^{\circ})) \leq \underline{PR}_1(x_1^*, \pi_\alpha^{\circ})$$

so that neither the high cost not the low cost firm 1 wants to deviate from x_1^* to x_1.

The continuum of equilibria as a function of α is illustrated in figure 7.



THE SET OF POOLING EQUILIBRIA

FIGURE 7

It has to be noted that the lower bound of the range of π_α° values which support a location $x_1^* \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}]$ as a PE is strictly positive. It follows from the inequality (52) and the fact that $\max \{ \bar{\pi}_\alpha(x_1), \underline{\pi}_\alpha(x_1) \}$ is strictly positive over the interval $[\underline{x}_1^{(-)}, \bar{x}_1^{(+)}]$.

Hence :

Proposition 3

Whatever the location $x_1 \in [\underline{x}^{(-)}_1, \bar{x}^{(+)}_1]$ there exist some priors $\pi_\alpha{}^0$ and some Bayesian posterior belief function $\pi^*(\,./x_1)$ which support it as a pooling equilibrium.

5.2 The Set of Pooling Equilibria Surviving Kreps' Intuitive Criterion.

As it will become clear in the following demonstrations the fact that $\underline{x}^{(-)}_1$ be lower, equal or higher than $\underline{x}^{(-)}_1$ (1/2, 1) plays a crucial role. The limit value α^k is that value for which :

$$\alpha < (=;>)\,\alpha^k \Rightarrow \underline{x}^{(-)}_1 > (=;<)\,\bar{x}^{(-)}_1 \quad (1/2, 1)$$

The different cases are discussed and illustrated in Appendix III.

5.2.1) The case where a separating equilibrium exists : $\alpha \in [0, (9/\sqrt{5}) - 4]$.

We show that the only PE, surviving KIC are the locations $x_1^* \in [\underline{x}_1(1), \bar{x}^{(+)}_1]$ when sustained by $\pi_\alpha{}^0 = 1$.

Remember first that for $\alpha \in (0, (9/\sqrt{5}) - 4]$ the $\overline{PR}_1(.)$ and $\underline{PR}_1(.)$ functions are such that $\bar{x}^{(+)}_1 \leqslant \underline{x}^{(+)}_1$ (with the strict inequality if $\alpha < (9/\sqrt{5})-4)$) ; second that $\underline{PR}_1(x_1, 1)$ is strictly decreasing on $[\bar{x}^{(+)}_1, \underline{x}^{(+)}_1]$ implying that $\underline{PR}_1(\bar{x}^{(+)}_1, 1) > \underline{PR}_1(\bar{x}^{(+)}_1, \pi_\alpha(\bar{x}^{(+)}_1))$ (or equivalently that $\pi_\alpha(\bar{x}^{(+)}_1) < 1$).

i) Consider any PE $(x_1^*, \pi_\alpha{}^0)$ with $\pi_\alpha{}^0 < 1$, so that $x_1^* < \underline{x}^{(+)}_1$.

Suppose first that $\alpha < (9/\sqrt{5}) - 4$. From $\pi^0{}_\alpha < 1$ it follows that $\underline{x}^{(+)}_1(x_1^*, \pi_\alpha{}^0) \in (\bar{x}^{(+)}_1, \underline{x}^{(+)}_1]$. So take any $\dot{x}_1 \in (\bar{x}^{(+)}_1, \underline{x}^{(+)}_1(x_1^*, \pi_\alpha{}^0))$. One gets :

$$\underline{PR}_1(\dot{x}_1, 1) > \underline{PR}_1(x_1^*, \pi_\alpha{}^0)$$

and

$$\overline{PR}_1(\dot{x}_1, \pi_\alpha) < \overline{PR}_1(x_1^*, \pi_\alpha{}^0) \qquad \forall\, \pi_\alpha$$

hence the perfectly identified low cost firm 1 is earning more profits at $\dot{x}_1$ than at x_1^* and whatever the posterior beliefs of firm 2, the high cost firm 1 is earning strictly less than its equilibrium profits. Hence x_1^* is eliminated by KIC.

If now $\alpha = (9/\sqrt{5}) - 4$, take $\dot{x}_1 = \bar{x}^{(+)}_1$, so that :

$$\underline{PR}_1 (\dot{x}_1 , 1) > \underline{PR}_1 (x_1^*, \pi_\alpha{}^0)$$

$$\overline{PR}_1 (\dot{x}_1 , \pi_\alpha) \leqslant \overline{PR}_1 (x_1^*, \pi_\alpha{}^0) \qquad \forall \pi_\alpha \quad (< \text{iff } \pi_\alpha < 1).$$

So, we conclude.

ii) Consider now any PE $(x_1^*, \pi_\alpha{}^0)$ with $\pi_\alpha{}^0 = 1$.

Suppose first that $x_1^* \leqslant \bar{x}_1 (1)$. We know from (39) - (40) that $x_1^* < \bar{x}^{(+)}_1 (x_1^*, 1) < \bar{x}^{(+)}_1$ and that the interval $(\bar{x}^{(+)}_1 (x_1^*, 1), \min \{\underline{x}^{(+)}_1 (x_1^*, 1), \bar{x}^{(+)}_1\})$ is not degenerate. For any $\dot{x}_1$ in that interval :

$$\underline{PR} (\dot{x}_1 , 1) > \underline{PR}_1 (x_1^*, 1)$$

$$\underline{PR}_1 (\dot{x}_1 , \pi_\alpha) < \overline{PR}_1 (x_1^*, 1) \qquad \forall \pi_\alpha$$

and KIC eliminates x_1^*.

Suppose second that $x_1^* \in (\bar{x}_1 (1), \underline{x}_1 (1))$. In this case from (41) to (43) we get :

$\forall \dot{x}_1 > x_1^*, \quad \forall \pi_\alpha :$

$$\underline{PR}_1 (x_1 , \pi_\alpha) < \overline{PR}_1 (x_1^*, 1)$$

and the interval $(x_1^*, \underline{x}^{(+)}_1 (x_1^*, 1))$ is not degenerate so that :

$\forall \dot{x}_1 \in (\dot{x}_1 , x^{(+)} (x_1^*, 1))$

$$\underline{PR}_1 (\dot{x}_1 , 1) > \underline{PR}_1 (x_1^*, 1)$$

and KIC eliminates x_1^*.

Suppose third that $x_1^* = \underline{x}_1 (1)$. Since $x_1^* = \underline{x}_1 (1)$ no deviation could give to the low cost firm 1 better than equilibrium profits. It is clearly the same for the high cost firm 1 and this PE survives KIC.

Suppose fourth that $x_1^* \in (\underline{x}_1 (1), \bar{x}^{(+)}_1]$. Then from (44) – (45) it follows that for any deviation which would improve the profits of the low cost firm 1 if rightly identified (a deviation to the left) there also exists a whole range of posteriors π_α for which the high cost firm 1 could also improve. Therefore these PE survive KIC.

5.2.2) The case $\alpha \in ((9/\sqrt{5}) - 4 , \alpha^k]$

Consider the set $X_{1\,\alpha} = [\underline{x}_1^{(-)} , \bar{x}_1^{(+)}] = [\underline{x}_1^{(-)} , 1/2]$ of PE, and define its partition $(X_{1\,\alpha}^{-} , X_{1\,\alpha}^{+})$ as follows :

$$x_1 \in X_{1\,\alpha}^{-} \Rightarrow \bar{\pi}^{\alpha}(x_1 , 1/2 , 1) \leqslant \max \{\bar{\pi}_{\alpha}(x_1) , \underline{\pi}_{\alpha}(x_1)\}$$

$$x_1 \in X_{1\,\alpha}^{+} \Rightarrow \bar{\pi}^{\alpha}(x_1 , 1/2 , 1) > \max \{\bar{\pi}_{\alpha}(x_1) , \underline{\pi}_{\alpha}(x_1)\}$$

The first sub-set $X_{1\,\alpha}^{-}$ is non-empty when $\alpha < \alpha^k$ since it contains at least some interval $[\underline{x}_1^{(-)} , \underline{x}_1^{(-)} + \varepsilon)$, $\varepsilon > 0$; the second sub-set $X_{1\,\alpha}^{+}$ is also non-empty since both $\underline{x}_1(0)$ and $\bar{x}_1^{(+)} = 1/2$ pertain to it.

i) The sub-case $x_1^{*} \in X_{1\,\alpha}^{+}$.

We show that no PE $(x_1^{*} , \pi_{\alpha}^{\,0})$, $\bar{\pi}_{\alpha}(x_1^{*} , 1/2 , 1) \leqslant \pi_{\alpha}^{\,0}$, survives KIC except those pertaining to $[\underline{x}_1(1) , 1/2]$ for $\pi_{\alpha}^{\,0} = 1$, and that any PE $(x_1^{*} , \pi_{\alpha}^{\,0})$, $\pi_{\alpha}^{\,0} < \bar{\pi}^{\alpha}(x_1^{*} , 1/2 , 1)$ survives (hence $\forall\ \pi_{\alpha}^{\,0}$ supporting $x_1^{*} = 1/2$ as a PE, $(1/2 , \pi_{\alpha}^{\,0})$ survives).

Let $(x_1^{*} , \pi_{\alpha}^{\,0})$, $\pi_{\alpha}^{\,0} \in [\bar{\pi}_{\alpha}(x_1^{*} , 1/2 , 1), 1)$ and $x_1^{*} \in [\underline{x}_1(1) , \bar{x}_1^{(+)}]$ then :

. either $\pi_{\alpha}^{\,0} > \bar{\pi}^{\alpha}(x_1^{*} , 1/2 , 1)$ and :

$$\bar{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}) < \underline{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}) < 1/2$$

. or $\pi_{\alpha}^{\,0} = \bar{\pi}^{\alpha}(x_1^{*} , 1/2 , 1)$ and :

$$\bar{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}) < \underline{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}) = 1/2 \text{ , but}$$

$$\bar{\pi}^{\alpha}(1/2 , x_1^{*} , \pi_{\alpha}^{\,0}) = 1 \text{ and } \underline{\pi}^{\alpha}(1/2 , x_1^{*} , \pi_{\alpha}^{\,0}) < 1$$

If $\pi_{\alpha}^{\,0} > \bar{\pi}^{\alpha}(x_1^{*} , 1/2 , 1)$ take any $\dot{x}_1 \in (\bar{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}) , \underline{x}_1^{(+)}(x_1^{*} , \pi_{\alpha}^{\,0}))$;

we get

$$\underline{PR}_1(\dot{x}_1 , 1) > \underline{PR}_1(x_1^{*} , \pi_{\alpha}^{\,0})$$

$$\overline{PR}_1(\dot{x}_1 , \pi_{\alpha}) < \overline{PR}_1(x_1^{*} , \pi_{\alpha}^{\,0}) \quad \forall\ \pi_{\alpha}$$

so that such a PE does not survives KIC.

If $\pi_\alpha^{\,0} = \bar{\pi}^\alpha\ (x_1^*, 1/2, 1)$, take $\dot{x}_1 = 1/2$. Then :

$$\underline{PR}_1\ (1/2, 1) > \underline{PR}_1\ (x_1^*, \pi_\alpha^{\,0})$$

$$\overline{PR}_1\ (\dot{1}/2, \pi_\alpha) < \overline{PR}_1\ (x_1^*, \pi_\alpha^{\,0}) \quad \forall\ \pi_\alpha$$

and such a PE is eliminated.

Let $(x_1^*, \pi_\alpha^{\,0})$, $x_1^* \in [\underline{x}_1\ (1), \bar{x}_1^{(+)}]$, $\pi_\alpha^{\,0} = 1$. The same argument as in 6.2.1 (ii) show that such a PE survives.

Let finally $(x_1^*, \pi_\alpha^{\,0})$, $\pi_\alpha^{\,0} < \bar{\pi}^\alpha\ (x_1^*, 1/2, 1)$. In such a case no deviation could either drive the high cost firm 1's profits above their equilibrium level if correctly identified, or reduce it down under its equilibrium level for all values of π_α if such a deviation is improving the low cost firm 1's profits when properly identified. Hence this kind of PE also survives KIC.

ii) The sub-case $x_1^* \in X_{1\alpha}^-$.

In order that such an x_1^*, be a PE it must be the case that $\pi_\alpha^{\,0} \geqslant \bar{\pi}^\alpha\ (x_1^*, 1/2, 1)$ and the same argument as in the preceeding paragraph for the same values of π_α eliminate $(x_1^*, \pi_\alpha^{\,0})$.

5.2.3) The case $\alpha \in (\alpha^k, 1/2]$

Note that $X_{1\alpha}^-$ now disappears so that $X_{1\alpha}^+ = [\underline{x}_1^{(-)}, 1/2]$.

$X_{1\alpha}^+$ can be split in two subsets :

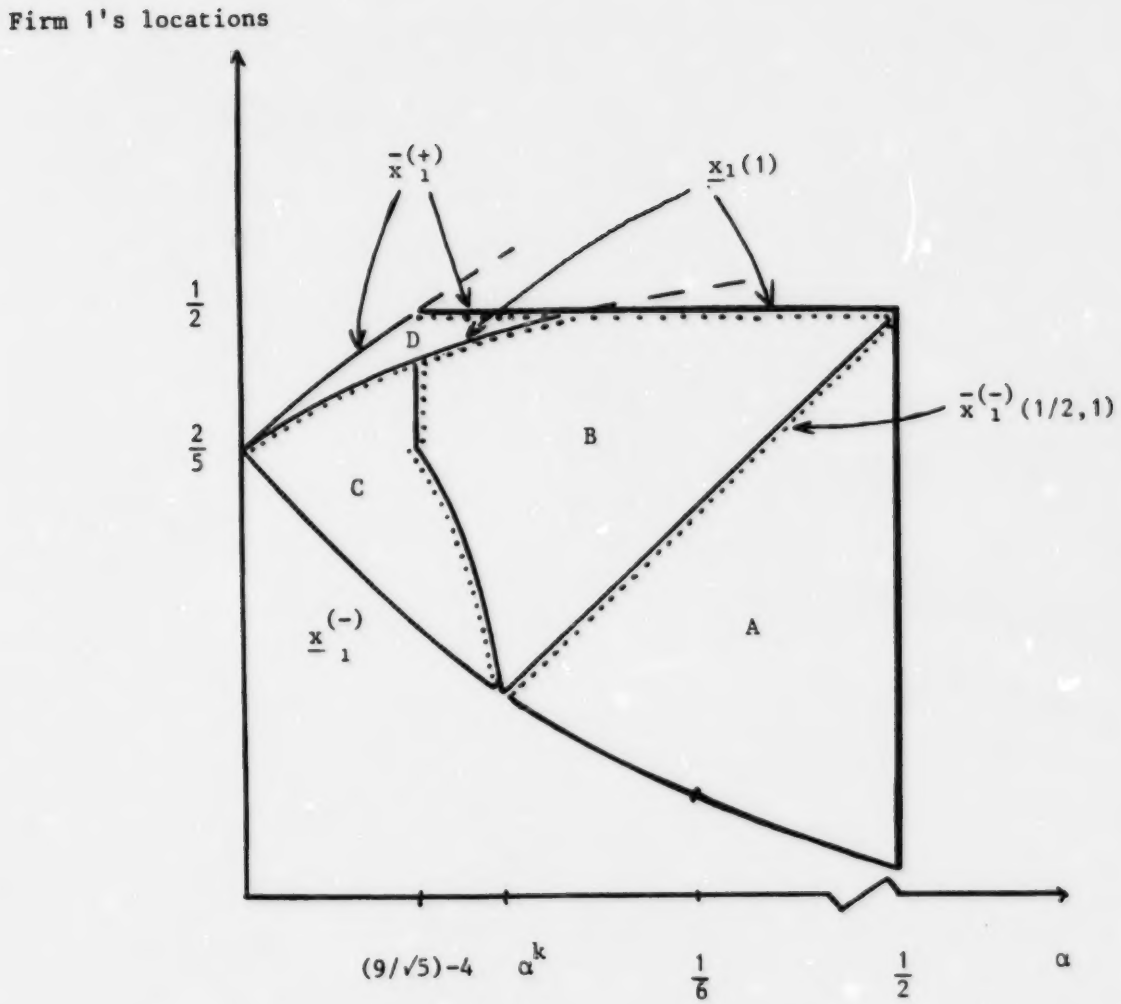
$$A_{1\alpha} = [\underline{x}_1^{(-)}, \bar{x}_1^{(-)}\ (1/2, 1)]$$

$$B_{1\alpha} = X_{1\alpha}^+ \setminus A_{1\alpha}$$

i) Let any PE$(x_1^*, \pi_\alpha^{\,0})$, $x_1^* \in A_{1\alpha}$. This PE is surviving KIC. It is a immediate consequence from the fact that no deviation of the high cost firm 1 can either give him higher than equilibrium profits if properly identified, or lower than equilibrium ones whatever the firm 2's posterior beliefs if such a deviation is giving to the properly identified low cost firm 1 higher than equilibrium profits.

ii) Let (x_1^*, π_α^0) a PE with , $x_1^* \in B_{1\alpha}$. If either $x_1^* < \underline{x}_1(1)$ and $\bar{\pi}^\alpha(x_1^*, 1/2, 1) \leqslant \pi_\alpha^0 \leqslant 1$, or $x_1^* \in [\underline{x}_1(1), 1/2]$ and $\bar{\pi}^\alpha(x_1^*, 1/2, 1) \leqslant \pi_\alpha^0 < 1$, then (x_1^*, π_α^0) is eliminated ; if either $\pi_\alpha^0 < \bar{\pi}^\alpha(x_1^*, 1/2, 1)$ or $x_1^* \in [\underline{x}, (1), 1/2]$ and $\pi_\alpha^0 = 1$, then (x_1^*, π_α^0) survives. Note that if $(1/2, \pi_\alpha^0)$ is a pooling equilibrium, it survives.

The correspondence of pooling equilibria, robust and not, as a function of α, is illustrated in figure 8.



Zone A : All the pooling equilibria survive

Zone B : For any x_1 there exists a set of π^0_e for which x_1 survives and a set of π^0_e for which it does not survive

Zone C : no pooling equilibrium survive

Zone D : All the pooling equilibria survive but not generically

The frontier between X and Y belongs to Y

THE SET OF POOLING EQUILIBRIA UNDER KIC.

FIGURE 8

5.3 The Set of Pooling Equilibria Surviving D1

We examine only the PE surviving KIC since those not surviving cannot survive D1 either.

i) Whatever $\alpha \in [0 , 1/2]$ consider first the PE $(x_1^*, \pi_\alpha{}^0)$ where $x_1^* \in [\underline{x}(1) , \bar{x}^{(+)}_1]$ and $\pi_\alpha{}^0 = 1$.

Let a deviation $\dot{x}_1$ to the left of x_1^*. Note first that :

$$\bar{x}^{(-)}_1 (x_1^*, 1) < \underline{x}^{(-)}_1 (x_1^*, 1)$$

and second that according to(4.4). (4.5)

$$\forall \dot{x}_1 \in [\bar{x}^{(-)}_1 (x_1^*, 1) , x_1^*) :$$

$$\bar{\pi}^\alpha (\dot{x}_1 , x_1^*, 1) < \underline{\pi}^\alpha (\dot{x}_1 , x_1^*, 1)$$

so that D1 implies that the firm is identified as a high cost firm. And therefore no firm would deviate.

As for deviations to the right of x_1^*, profits are reduced whatever the type and beliefs.

Hence such degenerate PE do survive.

ii) Whatever $\alpha \in ((9 / \sqrt{5}) - 4 , 1/2)$ any P E $(x_1^*, \pi_\alpha{}^0)$, $x_1^* \in X_1^+{}_\alpha \setminus \{1/2\}$ and $\pi_\alpha{}^0 < \bar{\pi}^\alpha (x_1^*, 1/2 , 1)$ is eliminated by D1.

Remember (46) to (51) and consider a deviation to the right $\dot{x}_1 = x_1^* + \varepsilon$, for ε small enough, we get :

$$\underline{\pi}^\alpha (\dot{x}_1 , x_1^* , \pi_\alpha{}^0) < \bar{\pi}^\alpha (\dot{x}_1 , x_1^* , \pi_\alpha{}^0).$$

Hence

$$\bar{D} (\dot{x}_1) \cup \bar{D}^0 (\dot{x}_1) \subset \underline{D} (\dot{x}_1)$$

implying the result.

iii) Whatever $\alpha \in ((9 / \sqrt{5}) - 4 , 1/2)$, if $x_1^* = 1/2$ and π_α^0 such that $(1/2 , \pi_\alpha^0)$ is a PE , $(1/2 , \pi_\alpha^0)$ is surviving.

Concerning deviations to the left, the argument is as in (i) and no deviation to the right is "feasible".

iv) Whatever $\alpha \in (\alpha^k , 1/2]$ any PE (x_1^* , π_α^0) , $x_1^* \in A_{1\alpha}$, is eliminated.

$x_1^* \in A_{1\alpha}$ implies $x_1^* < \underline{x}(\pi_\alpha^0)$, so take $\dot{x}_1 = x_1^* + \varepsilon$ and proceed as in (ii).

Noting that for any $\alpha \in ((9 / \sqrt{5}) - 4 , 1/2]$, $\max (\bar{\pi}_\alpha(1/2), \underline{\pi}_\alpha(1/2)) = \bar{\pi}_\alpha(1/2)$ (see appendix IV) we conclude :

Proposition 4

For any $\alpha \in (9 / \sqrt{5} - 4 , 1/2]$ the only non degenerate pooling equilibrium surviving D1 are :

$(1/2 , \pi_\alpha^0)$ **for** $\pi_\alpha^0 \in [\bar{\pi}_\alpha(1/2), 1]$, $\forall \alpha \in ((9 / \sqrt{5}) - 4 , 1/2]$.

6. SEMI-SEPARATING EQUILIBRIA (SSE)

For the separating and pooling equilibria it was possible to consider only pure strategy equilibria. For semi-pooling or equivalently semi-separating equilibria it is not possible by definition, but we will restrict ourselves to the least mixing strategies in the following sense. Firm 1 will be assumed to choose some location $\hat{x}_1(c_1)$ with certainty for one of its types c_1, and to mix between $\hat{x}_1(c_1') \neq \hat{x}_1(c_1)$, $c_1' \neq c_1$, with probability μ and $\hat{x}_1(c_1)$ with probability $1 - \mu$. Firm 2's strategy is a pure strategy. Again for the domain in the (π_α^0, α)-space where such equilibria exist, by the Cho and Sobel uniqueness theorem, we will capture all the equilibria surviving D1.

With such kind of strategies we distinguish those SSE where the high cost firm 1 randomizes and those where it is the low cost firm. Concerning the first type there exists a whole range of common locations of the high and low cost firm 1 the measure of which is first increasing from $\alpha = 0$ to $\alpha = (9/\sqrt{5}) - 4$ and then decreasing up to $\alpha = 1/2$. The high cost firm 1 when revealing its type is always locating at $\bar{x}_1(0)$, the location maximizing its profits when identified. Note that for such equilibria to exist the prior beliefs must put sufficient weight on the high cost firm type. The only equilibria of this kind which survive KIC are those obtained for $\alpha \in ((9/\sqrt{5}) - 4, 1/2)$ and amongst them the only ones surviving D1 are those where the low cost firm is located at the center of the market. Concerning now the SSE where the low cost firm 1 randomizes there exists for $\alpha \in (0, 1/8]$ a continuum of common locations of both the high and low cost firm 1, the measure of which is first increasing for $\alpha \in (0, (9/\sqrt{5}) - 4]$ and then decreasing towards zero. For $\alpha = 1/8$ this equilibrium is unique. If $\alpha > 1/8$ no such equilibrium exists. None of these equilibria survives KIC hence a fortiori D1.

6.1 The Set of Semi-Separating Equilibria

6.1.1) The set of semi-separating equilibria in which the high cost firm randomizes.

In such equilibria the low cost firm is always playing $\hat{x}_1^*(c - \alpha)$ and the high cost one is playing either $\hat{x}_1^*(c)$ with probability μ^* or $\hat{x}_1^*(c - \alpha)$ with probability $1 - \mu^*$. We will denote $\hat{x}_1^*(c - \alpha)$ by $\underline{x}_1^*$ and $\hat{x}_1^*(c)$ by $\bar{x}_1^*$. Hence by Bayes' rule the high cost firm 1 is rightly identified when playing $\bar{x}_1^*$; also the posterior beliefs of the firm 2 observing $\underline{x}_1^*$ must be :

$$\pi^* (c - \alpha / \underline{x}_1^*) = \frac{\pi_\alpha^\circ}{\pi_\alpha^\circ + (1 - \mu^*)(1 - \pi_\alpha^\circ)}$$

with $\pi^* (c - \alpha / \underline{x}_1^*) \in (0 , 1)$ if $\pi_\alpha^\circ \in (0 , 1)$ and $\mu^* \in (0 , 1)$, a condition which must be satisfied in a SSE.

An argument similar to the argument given in the study of S E would show that $\bar{x}_1^* = \bar{x}_1(0)$ the location maximising the profits of the high cost firm 1 correctly identified ; hence the only problem is concerning the location $\underline{x}_1^*$.

In order to describe the whole range of possible $\underline{x}_1^*$ let us observe that for any given value $\pi_\alpha^* \in (0 , 1)$ of the posterior beliefs, $\pi^* (c - \alpha / \underline{x}_1^*)$ can be made Bayesian in an infinity of ways since the equation :

$$\pi_\alpha^* = \frac{\pi_\alpha^\circ}{\pi_\alpha^\circ + (1 - \mu^*)(1 - \pi_\alpha^\circ)}$$

has an infinity of solutions $(\pi_\alpha^\circ , \mu^*)$, $\pi_\alpha^\circ \in (0 , 1)$, $\mu^* \in (0 , 1)$. Hence we conduct the discussion directly in terms of the permitted values of $\pi^* (c - \alpha / \underline{x}_1^*)$, which will be denoted by $\pi^*_\alpha(\underline{x}_1^*)$, going back to the primitive parameters π_α° and μ^* only in fine (see also section 8). Let us examine the constraints put by the high cost and the low cost firm 1.

In order that the high cost firm 1 be mixing between $\bar{x}_1(0)$ and $\underline{x}_1^*$ we must have :

$$\overline{PR}_1 (\bar{x}_1(0) , 0) = \overline{PR}_1 (\underline{x}_1^* , \pi^*_\alpha (\underline{x}_1^*))$$

A necessary condition for this equality to hold is that :

$$\underline{x}_1^* \in [\bar{x}_1^{(-)} , \bar{x}_1^{(+)}] \text{ and } \pi^*_\alpha (\underline{x}_1^*) = \bar{\pi}_\alpha (\underline{x}_1^*).$$

Furthermore, in order that this high cost firm does not deviate from $\{\bar{x}_1(0) , \underline{x}_1^*\}$ the posterior beliefs for any off-equilibrium location $x_1 \notin \{\bar{x}_1(0) , \underline{x}_1^*\}$ must be such that :

$$\pi^* (c - \alpha / x_1) \leqslant \bar{\pi}_\alpha (x_1) \tag{53}$$

As for the low cost firm, the first constraint is that

$$\underline{PR}_1 (\underline{x}_1^* , \pi^*_\alpha (\underline{x}_1^*)) \geqslant \underline{PR}_1 (\underline{x}_1(0) , 0)$$

since otherwise it would deviate at $\underline{x}_1(0)$ whatever the beliefs of firm 2 observing $\underline{x}_1(0)$; but this implies that :

$$\pi^*_\alpha(\underline{x}_1^*) \geq \Pi_\alpha(\underline{x}_1^*) \tag{54}$$

Furthermore for any $x_1 \neq \underline{x}_1^*$ the posterior beliefs must be such that :

$$\pi^*(c - \alpha / x_1) \leq \Pi^\alpha(x_1, \underline{x}_1^*, \pi^*_\alpha(\underline{x}_1^*))$$

Taking into account the two constraints (53) and (54) we conclude that $\underline{x}_1^*$ is restricted to the interval of locations x_1 for which $\Pi_\alpha(x_1) \leq \bar{\pi}_\alpha(x_1)$. It is shown in Appendix IV that such an interval is equal to $[(\alpha^2 + 6\alpha + 4)^{1/2} / 5 , \bar{x}^{(+)}_1]$ for $\alpha \in (0 , (9/\sqrt{5}) - 4, 1/6]$; to $[(\alpha^2 + 6\alpha + 4)^{1/2} / 5 , 1/2]$ for $\alpha \in ((9/\sqrt{5}) - 4 , 1/6]$; and to $[7920\,\alpha^2 - 240\,\alpha + 1)^{1/2} / 180\,\alpha, 1/2]$ for $\alpha \in [1/6 ; 1/2]$.

Last observe that $\underline{x}_1^*$ cannot be equal to $\bar{x}^{(+)}_1$ when $\overline{PR}_1(\bar{x}^{(+)}_1, 1) = \overline{PR}_1(\underline{x}_1(0), 0)$ since it would imply that $\pi^*_\alpha(\underline{x}_1^*) = 1$. Hence :

Proposition 5

For $\alpha \in (0 , (9/\sqrt{5}) - 4]$ there exists a continuum of SSE :

$$\{ \hat{x}_1^*(c) = \bar{x}_1(0), \ \hat{x}_1^*(c - \alpha) \in [(\alpha^2 + 6\alpha + 4)^{1/2} / 5 , \bar{x}^{(+)}_1] \} ;$$

for $\alpha \in ((9/\sqrt{5}) - 4 , 1/6]$, a continuum :

$$\{ \hat{x}_1^*(c) = \bar{x}_1(0), \ \hat{x}_1^*(c - \alpha) \in [(\alpha^2 + 6\alpha + 4)^{1/2} / 5 , \bar{x}^{(+)}_1] \}$$

and at last for $\alpha \in [1/6, 1/2]$, a continuum :

$$\{ \hat{x}_1^*(c) = \bar{x}_1(0), \ \hat{x}_1^*(c - \alpha) \in [(7920\,\alpha^2 - 240\,\alpha + 1)^{1/2} / 180\,\alpha, 1/2] ;$$

each one supported by the set of prior beliefs $\pi_\alpha{}^0$ and probabities μ^* (with which the high cost firm 1 is playing $\bar{x}_1(0)$) such that :

$$\bar{\pi}^*(\hat{x}_1^*(c - \alpha)) = \frac{\pi_\alpha{}^0}{\pi_\alpha{}^0 + (1 - \mu^*)(1 - \pi_\alpha{}^0)}$$

and by any posterior belief function $\pi^*(c - \alpha / x_1)$ such that :

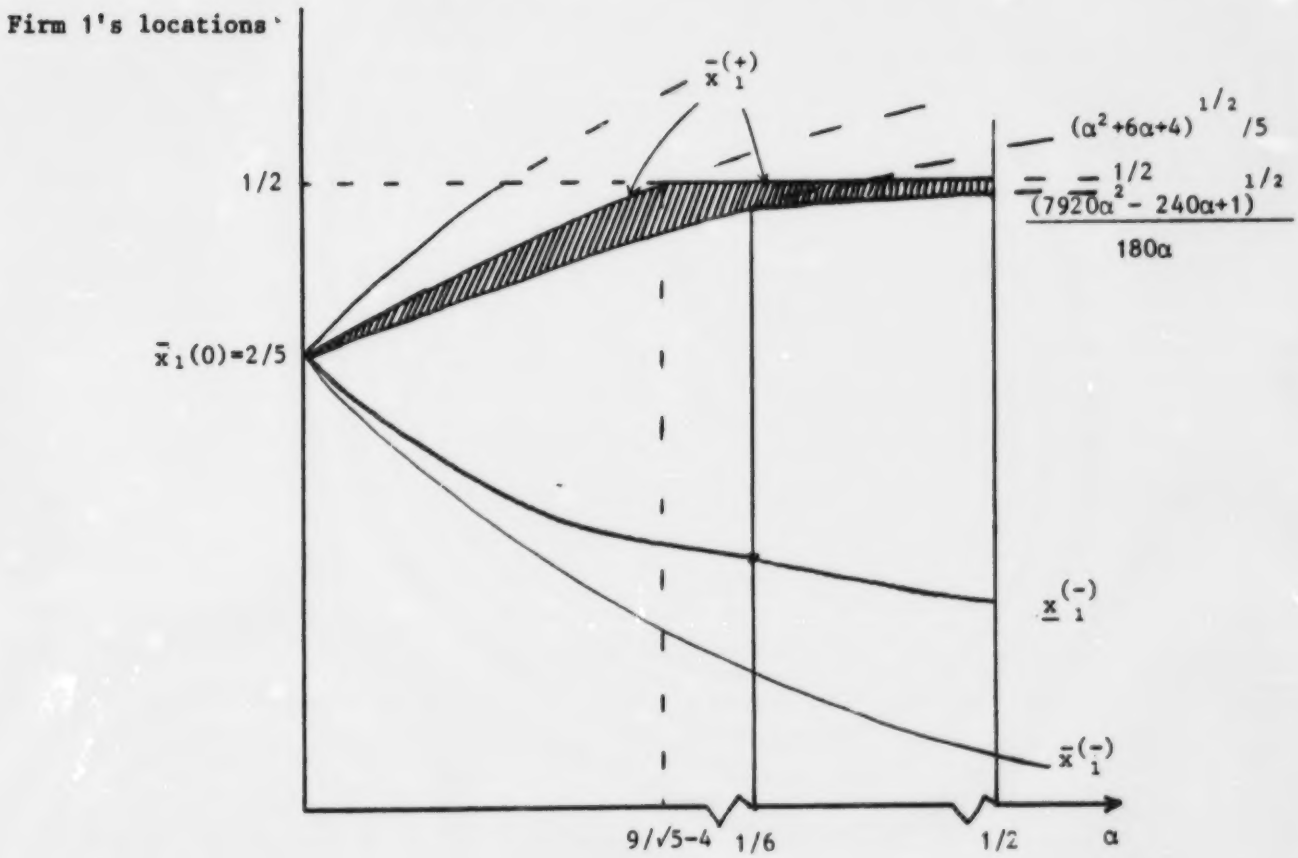
$$\pi^* \ (c - \alpha \ / \ \bar{x}_1 (0)) = 0$$

$$\pi^* \ (c - \alpha \ / \ \hat{x}_1^* \ (c - \alpha \ )) = \bar{\pi}_\alpha \ (\hat{x}_1^* \ (c - \alpha \ ))$$

and for $x_1 \in \{ \ \bar{x}_1 (0) \ , \ \hat{x}_1^* \ (c - \alpha \ )) \ :$

$$\pi^* \ (c - \alpha \ / \ \hat{x}_1^*) < \min \{ \ \bar{\pi}_\alpha \ (x_1) \ , \ \Pi^\alpha \ (x_1 \ , \ \hat{x}_1^* \ (c - \alpha) \ , \ \bar{\pi}_\alpha \ (\hat{x}_1^* \ (c - \alpha)))\}$$

The correspondence giving the set of equilibrium common locations of the high and low cost firm 1 is illustrated in figure 9 :



THE SET OF SSE IN WHICH THE HIGH COST FIRM RANDOMIZES.

FIGURE 9

6.1.2) The set of semi-separating equilibria in which the low cost firm randomizes.

In these SSE the high cost firm 1 is always playing $\hat{x}_1^*(c) = \bar{x}_1^*$, whereas the low cost is playing either $\hat{x}_1^*(c-\alpha) = \underline{x}_1^*$ with probability μ^* or $\bar{x}_1^*$ with probability $1-\mu^*$ so that by the Bayes rule first the low cost firm 1 is clearly identified when playing $\underline{x}_1^*$ and second the posterior beliefs of firm 2 observing $\bar{x}_1^*$ are :

$$\bar{\pi}^*(c-\alpha / \bar{x}_1^*) = \frac{\pi_\alpha^0 (1-\mu^*)}{\pi_\alpha^0 (1-\mu^*) + (1-\pi_\alpha^0)} .$$

Following a similar way of reasoning as for the SSE identifying the low cost firm, we construct the set of this kind of SSE by working first directly on $\pi^*_\alpha(\bar{x}_1^*)$, the Bayesian posterior beliefs of firm 2 observing $\bar{x}_1^*$.

Consider first the low cost firm. Note that we must have :

$$\underline{x}_1^* \in [\underline{x}_1^{(-)}, \underline{x}_1^{(+)}] \tag{55}$$

since $\underline{x}_1^*$ outside this interval would imply that $\underline{x}_1(0)$ is a strictly better location whatever the posterior beliefs of firm 2.

Now because firm 1 is mixing between $\underline{x}_1^*$ and $\bar{x}_1^*$:

$$\underline{PR}_1(\underline{x}_1^*, 1) = \underline{PR}_1(\bar{x}_1^*, \pi^*_\alpha(\bar{x}_1^*))$$

so that :

$$\bar{x}_1^* \in [\underline{x}_1^{(-)}(\underline{x}_1^*, 1), \underline{x}^{(+)}(\underline{x}_1^*, 1)] \text{ and } \pi^*_\alpha(\bar{x}_1^*) = \pi^\alpha(\bar{x}_1^*, \underline{x}_1^*, 1). \tag{56}$$

Finally in order that it does not deviate from $x_1 \in \{\underline{x}_1^*, \bar{x}_1^*\}$ to any other $x_1' \notin \{\underline{x}_1^*, \bar{x}_1^*\}$ the posterior beliefs of firm 2 after observing such locations must verify :

$$\pi^*(c-\alpha / x_1') \leq \pi^*(x_1', \underline{x}_1^*, 1). \tag{57}$$

Let us now examine the high cost firm. In order that $\bar{x}_1^*$ be not dominated by $\bar{x}_1(0)$ we must first have :

$$\underline{x}_1^* \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}] \text{ and } \pi^*_\alpha(\bar{x}_1^*) \geq \bar{\pi}_\alpha(\bar{x}_1^*). \tag{58}$$

Second in order that the high cost firm 1 does not deviate from $\bar{x}_1^*$ to $\underline{x}_1^*$ the following condition must hold :

$$(59) \qquad \bar{\pi}^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1) \leqslant \pi^*_{\alpha}(\bar{x}_1^*) = \underline{\pi}^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1)$$

and finally in order it does not deviate from $\bar{x}_1^*$ to any x_1 the posterior beliefs must be such that :

$$(60) \qquad \pi^*(c - \alpha / x_1) \leqslant \bar{\pi}^{\alpha}(x_1, \bar{x}_1^*, \underline{\pi}^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1)).$$

Note that this condition is met at $x_1 = \underline{x}_1^*$ if (59) itself is met.

Any pair $\{\bar{x}_1^*, \underline{x}_1^*\}$ satisfying the six conditions (55) to (60), may be sustained as an SSE. We show next that for $\alpha < 1/8$ there exists a continuum of such SSE whereas for $\alpha \geqslant 1/8$ no such SSE exists. This bound comes from the fact that this kind of SSE exists only if $\underline{x}_1^*$ is in the interval $(\underline{x}_1(1), \underline{x}_1^{(+)}]$ which disappears for $\alpha \geqslant 1/8$ (for $\alpha \geqslant 1/8$, $\underline{x}_1(1) = 1/2$).

i) Suppose $\alpha < 1/8$ and take $\underline{x}_1^* \in (\underline{x}_1(1), \underline{x}_1^{(+)}]$. We must distinguish according to either $\underline{x}_1^* \in (\underline{x}_1(1), \bar{x}_1^{(+)}]$ or $\underline{x}_1^* \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}]$ (if the value of α is such that this last interval exists, i.e if $\alpha < (9/\sqrt{5}) - 4 < 1/8$).

i.a) $\underline{x}_1^* \in (\underline{x}_1(1), \bar{x}_1^{(+)}]$

In this case $\underline{x}_1^{(+)}(\underline{x}_1^*, 1) = \bar{x}_1^{(+)}(\underline{x}_1^*, 1) = \underline{x}_1^*$ and

$$(61) \qquad \bar{x}_1^{(-)}(\underline{x}_1^*, 1) < \underline{x}_1^{(-)}(\underline{x}_1^*, 1) < \underline{x}_1(1) \leqslant \underline{x}_1^*.$$

It has already been demonstrated (see Appendix II) that :

$$(62) \qquad \underline{x}_1^* \geqslant \underline{x}_1(1)$$

$$\Rightarrow \qquad \left. \frac{\partial \bar{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1)}{\partial x_1} \right|_{x_1 = x_1^*} > 0$$

and

$$\left. \frac{\partial \underline{\pi}^{\alpha}(x_1, x_1^*, 1)}{\partial x_1} \right|_{x_1 = \underline{x}_1^*} \geqslant 0$$

so, if $x_1 \in (\bar{x}_1^{(-)}(\underline{x}_1^*, 1), \underline{x}_1^*)$:

$$(63) \qquad \underline{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1) > \bar{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1)$$

and by definition if $x_1 \notin (\bar{x}_1^{(-)}(\underline{x}_1^*, 1), \underline{x}_1^*)$:

$$(64) \qquad \underline{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1) = \bar{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1) = 1$$

Furthermore, because $\underline{x}_1^* < \bar{x}_1^{(+)}$, $\bar{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1) > \bar{\pi}_{\alpha}(x_1)$. Hence, for any $\bar{x}_1^* \in [\underline{x}_1^{(-)}(\underline{x}_1^*, 1), \underline{x}_1^*]$, all the six conditions (55) to (60) can be met by the pair of locations $\{\bar{x}_1^*, \underline{x}_1^*\}$.

Observing that $\bar{x}_1^{(-)}(\underline{x}_1^*, 1) = 2\,\underline{x}_1(1) - \underline{x}_1^*$, we conclude as follows :

Proposition 6

For $\alpha \in (0, 1/8)$ there exists a continuum of SSE $\{\hat{x}_1^*(c - \alpha) \in (\underline{x}_1(1), \bar{x}_1^{(+)}], \hat{x}_1^*(c) \in [2\,\underline{x}_1(1) - \hat{x}_1^*(c - \alpha), \hat{x}_1^*(c - \alpha))\}$ each one supported by the set of prior beliefs π^0_{α} and probability μ^* (with which the low cost firm is playing $\hat{x}_1^*(c - \alpha)$) such that :

$$\underline{\pi}^{\alpha}(\hat{x}_1^*(c), \hat{x}_1^*(c - \alpha), 1) = \frac{\pi_{\alpha}^0 (1 - \mu^*)}{\pi_{\alpha}^0 (1 - \mu^*) + (1 - \pi_{\alpha}^0)}$$

and by any posterior belief function $\pi^*(c - \alpha / x_1)$ such that :

$$\pi^*(c - \alpha / \hat{x}_1^*(c - \alpha)) = 1$$
$$\pi^*(c - \alpha / \hat{x}_1^*(c)) = \underline{\pi}^{\alpha}(\hat{x}_1^*(c), \hat{x}_1^*(c - \alpha), 1)$$

and for $x_1 \notin \{\hat{x}_1^*(c), \hat{x}_1^*(c - \alpha)\}$:

$$\pi^*(c - \alpha / x_1) \leq \min\{\bar{\pi}^{\alpha}(x_1, \hat{x}_1^*(c), \underline{\pi}^{\alpha}(\hat{x}_1^*(c), \hat{x}_1^*(c - \alpha), 1)), \underline{\pi}^{\alpha}(x_1, \hat{x}_1^*(c - \alpha), 1)\}.$$

i.b) $\underline{x}_1^* \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}]$, implying $\alpha \in (0, (9/\sqrt{5}) - 4)$.

As in the preceding case all the (61), (62) and (63) conditions are also verified ; but now since $\underline{x}_1^* < \bar{x}_1^{(+)}$ there exists some interval $(\dot{x}_1, \underline{x}_1^*)$ on which $\bar{\pi}_{\alpha}(x_1) > \underline{\pi}^{\alpha}(x_1, \underline{x}_1^*, 1)$ so that $\bar{x}_1^*$ cannot be chosen in this range of locations.

Let us denote by $\dot{x}_1(\underline{x}_1^*)$ the solution of :

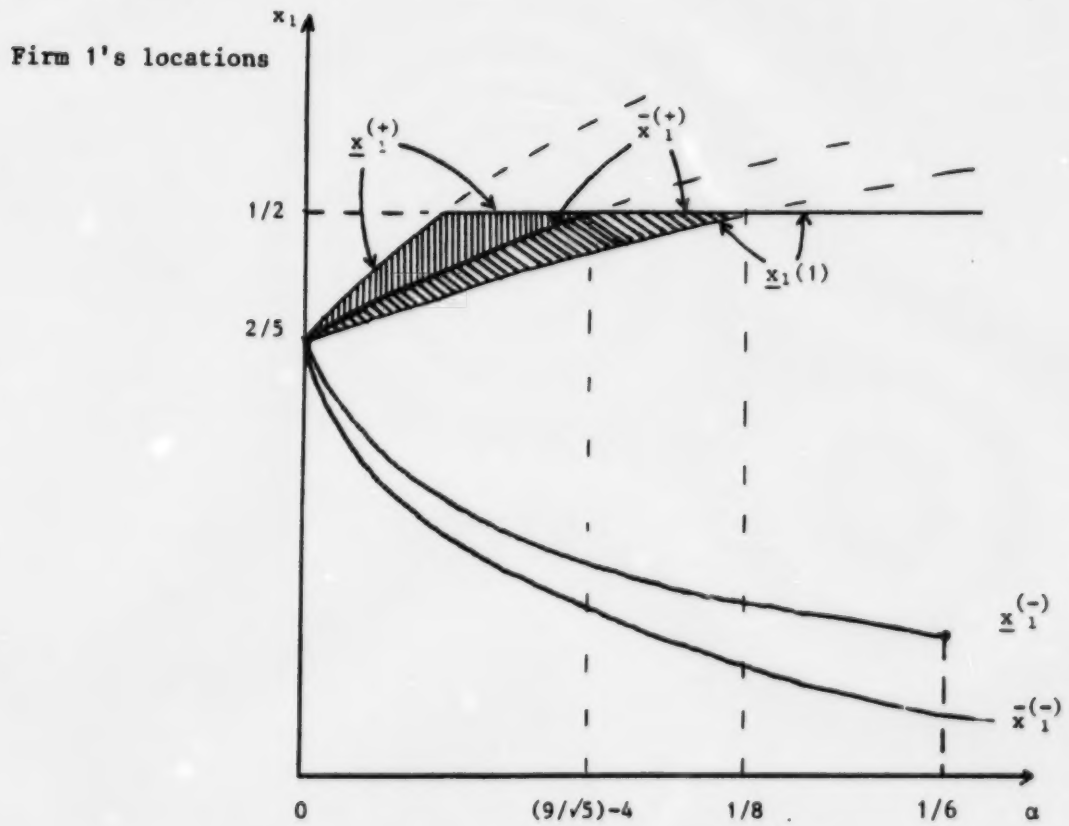
$$\Pi_\alpha(x_1) = \Pi^\alpha(x_1, \underline{x}_1^*, 1).$$

It is shown in Appendix V that if x_1 is inside $(\underline{x}_1^{(-)}(\underline{x}_1^*, 1), \dot{x}_1(\underline{x}_1^*))$ then $\bar{\pi}_\alpha(x_1) < \Pi^\alpha(x_1, \underline{x}_1^*, 1)$, so :

Proposition 7

For $\alpha \in (0, (9/\sqrt{5}) - 4)$ there exists a continuum of SSE $\{\hat{x}_1^*(c - \alpha) \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}], \hat{x}_1^*(c) \in [2\underline{x}_1(1) - \hat{x}_1^*(c - \alpha), \dot{x}_1(\underline{x}_1^*))\}$, each one supported by the same kind of set of prior beliefs π_α^0, probability μ^* and posterior belief functions as in the preceding case (i.a).

The first component of the equilibrium correspondence is illustrated at figure 10.



THE SET OF SSE IN WHICH THE LOW-COST FIRM 1 RANDOMIZES.

FIGURE 10

ii) Clearly x_1^* cannot be at $x_1(1)$ since then no other location could provide to the low cost firm 1 the same profits whatever firm 2's posterior beliefs, hence the impossibility of a firm 1 mixing between $x_1(1)$ and some another location. Next for $x_1^* < x_1(1)$ we have :

$$x_1^* = \bar{x}_1^{(-)}(x_1^*, 1) = x_1^{(-)}(x_1^*, 1) < x_1(1) \leqslant \bar{x}_1^{(+)}(x_1^*, 1) \leqslant x_1^{(+)}(x_1^*, 1)$$

but then (see Appendix II) :

$$\bar{\pi}^{\alpha}(x_1, x_1^*, 1) > \pi^{\alpha}(x_1, x_1^*, 1) \text{ if } x_1 \in (x_1^*, x_1^{(+)}(x_1^*, 1))$$

so that (60) is never satisfied in this case. We conclude that there exists no SSE in which the low cost firm 1 is rightly identified when playing an $x_1^* \leqslant x_1(1)$.

6.2. The Set of Semi-Separating Equilibria Surviving Kreps' Intuitive Criterion

Amongst the set of SSE the only sub set surviving Kreps' intuitive criterion includes those SSE in which the high cost firm 1 reveals its type by playing $x_1^* = \bar{x}_1(0)$ and only for the values of α such that there exist no SE, i.e for $\alpha \in ((9/\sqrt{5}) - 4, 1/2]$.

i) Consider first the case of SSE in which the high cost firm randomizes

i.a) $\alpha \in (0, (9/\sqrt{5}) - 4)$

In those SSE we have

$$\hat{x}_1^*(c - \alpha) = x_1^* \in [(\alpha^2 + 6\alpha + 4)^{1/2}/5, \bar{x}_1^{(+)})$$

and :

$$\pi^*(c - \alpha) / \hat{x}_1^*(c - \alpha)) = \pi^*_{\alpha}(x_1^*) = \bar{\pi}_{\alpha}(x_1^*)$$

It is clear (see Appendix V for an illustration) that $\pi^*(x_1, \bar{x}_1^{(+)}, 1) > \bar{\pi}_{\alpha}(x_1)$ over the interval $[(\alpha^2 + 6\alpha + 4)^{1/2}/5, \bar{x}_1^{(+)})$ for the implied values of α, so that :

$$\pi^{\alpha}(x_1^*, \bar{x}_1^{(+)}, 1) > \bar{\pi}_{\alpha}(x_1^*)$$

hence, since $\bar{x}_1^{(+)} < 1/2$ for $\alpha < (9/\sqrt{5}) - 4$:

$$\underline{x}_1^{(+)}(\underline{x}_1^*, \bar{\pi}_\alpha(\underline{x}_1^*)) > \bar{x}_1^{(+)}.$$

So for any $\dot{x}_1 \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}(\underline{x}_1^*, \bar{\pi}_\alpha(\underline{x}_1^*)))$:

$$\underline{PR}_1(\dot{x}_1, 1) > \underline{PR}_1(\underline{x}_1^*, \bar{\pi}_\alpha(\underline{x}_1^*))$$

and the firm 1's profit at $\dot{x}_1$ if rightly identified, are higher that the equilibrium profits. Now, since $\dot{x}_1 > \bar{x}_1^{(+)}$:

$$\forall \pi_\alpha : \overline{PR}_1(\dot{x}_1, \pi_\alpha) < \overline{PR}_1(\bar{x}_1(0), 0) = \overline{PR}_1(\underline{x}_1^*, \bar{\pi}_\alpha(x_1^*))$$

and whatever firm 2's beliefs the high cost firm 1's profits are lower than the equilibrium profits. Hence this kind of SSE is ruled out.

i.b) $\alpha \in [(9/\sqrt{5}) - 4, 1/2]$

The difference with the preceding case is that now $\bar{x}_1^{(+)} = 1/2$, and any deviation from $\underline{x}_1^*$ which would permit higher than equilibrium profits for the low cost firm 1 is also giving higher profits than equilibrium profits to the high cost firm 1 for some range of posterior beliefs. Since no deviation, neither from $\bar{x}_1(0) = \bar{x}_1^*$ nor from $\underline{x}_1^*$, could generate higher than equilibrium profits for the high cost firm if correctly identified, we conclude that this kind of SSE is surviving the Kreps criterion.

ii) Second let us examine the SSE in which the low cost randomizes.

In some of those equilibria, when $\alpha < (9/\sqrt{5}) - 4$, the low cost firm 1 plays sometimes $\underline{x}_1^* \in (\bar{x}_1^{(+)}, \underline{x}_1^{(+)}]$ and sometimes $\bar{x}_1^*$, obtaining the same profit in the two locations. Now taking $\dot{x}_1 \in (\bar{x}_1^{(+)}, \underline{x}_1^*)$ the high cost firm 1 is obtaining more than its equilibrium profits if rightly identified, whereas the low cost firm is obtaining less whatever firm 2's beliefs ; hence this kind of equilibrium is killed by the intuitive criterion.

In the other sub set of such SSE the low cost firm 1 plays sometimes $\underline{x}_1^* \in (\underline{x}_1(1), \bar{x}_1^{(+)}]$, and sometimes $\bar{x}_1^* \in [2\,\underline{x}_1(1) - \underline{x}_1^*, \underline{x}_1^*)$, and the high cost firm 1 always $\bar{x}_1^*$. Remember that at $\underline{x}_1^*$:

$$\Pi^{\alpha}(\bar{x}_1^*,\ \underline{x}_1^*,\ 1)\ >\ \bar{\pi}^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1)$$

so that :

$$\bar{x}^{(+)}_1(\bar{x}_1^*,\ \Pi^{\alpha}(\bar{x}_1^*,\ \underline{x}_1^*,\ 1)) < \bar{x}^{(+)}_1(\bar{x}_1^*,\ \bar{\pi}^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1)) = \underline{x}_1^*.$$

Hence take any $\dot{x}_1 \in (\bar{x}^{(+)}_1(\bar{x}_1^*, \Pi^{\alpha}(\bar{x}_1^*, \underline{x}_1^*, 1)),\ \underline{x}_1^*)$ we have first (since $\underline{PR}_1(.)$ is decreasing in x_1 on $(\underline{x}_1(1),\ \underline{x}_1^*)$) :

$$\underline{PR}_1(\dot{x}_1, 1) > \underline{PR}_1(\underline{x}_1^*, 1) = \underline{PR}_1(\bar{x}_1^*,\ \Pi^{\alpha}(\bar{x}_1^*,\ \underline{x}_1^*,\ 1))$$

and the low cost firm 1 rightly identified is earning, at $\dot{x}_1$, more than its equilibrium profits ; and second (since $\dot{x}_1 > \bar{x}^{(+)}_1(\bar{x}_1^*,\ \Pi^{\alpha}(\bar{x}_1^*,\ \underline{x}_1^*,\ 1))$:

$$\forall\, \pi_{\alpha}:\ \ \overline{PR}_1(\dot{x}_1, \pi_{\alpha}) < \overline{PR}_1(\bar{x}_1^*,\ \Pi^{\alpha}(\bar{x}_1^*,\ \underline{x}_1^*,\ 1))$$

and whatever firm 2's posterior beliefs the high cost firm 1 is earning, at $\dot{x}_1$, less than its equilibrium profits. This kind of equilibrium does not survive the intuitive criterion.

In conclusion there exists no SSE where the low cost firm 1 randomizes which survives the intuitive criterion.

6.3 The Set of Semi-Separating Equilibria Surviving D1

Consider the SSE surviving the intuitive criterion, i.e those in which the high cost firm 1 is sometimes revealing its type and $\alpha \in [(9/\sqrt{5}) - 4\ ,\ 1/2]$. For such SSE the equilibrium location always played by the low cost firm 1 and sometimes played by the high cost one is $\underline{x}_1^* \in [(\alpha^2 + 6\,\alpha + 4)^{1/2}/5,\ 1/2)$ for $\alpha \in [(9/\sqrt{5}) - 4\ ,\ 1/6]$ and $\underline{x}_1^* \in [(7920\ \alpha^2 - 240\ \alpha + 1)^{1/2}/180\ \alpha,\ 1/2)$ for $\alpha \in [1/6\ ,\ 1/2]$ and

$$\pi^*(c - \alpha / \underline{x}_1^*) = \bar{\pi}_{\alpha}(\underline{x}_1^*) < 1.$$

For $x_1 \in (\underline{x}_1^*,\ 1/2)$:

$$\bar{\pi}^{\alpha}(x_1\ ,\ \underline{x}_1^*,\ \bar{\pi}_{\alpha}(\underline{x}_1^*)) = \bar{\pi}_{\alpha}(\underline{x}_1^*) > \Pi^{\alpha}(x_1\ ,\ \underline{x}_1^*,\ \bar{\pi}_{\alpha}(\underline{x}_1^*))$$

so that :

$$\bar{D}(x_1) \cup \bar{D}^{0}(x_1) \subset \underline{D}(x_1)$$

and according to D1 firm 2 should infer that it is facing a low cost firm 1. Now $\underline{PR}_1(x_1\ ,\ \pi_{\alpha})$ is decreasing in x_1 on $(\underline{x}_1^*,\ 1/2)$ and $\bar{\pi}_{\alpha}(\underline{x}_1^*) < 1$. So there exists $\varepsilon > 0$ such that for $\dot{x}_1 \in (\underline{x}_1^*,\ \underline{x}_1^* + \varepsilon)$:

$$\underline{PR}_1(\dot{x}_1, 1) > \underline{PR}_1(x_1^*, \bar{\pi}_\alpha(x_1^*))$$

and the low cost firm 1 will deviate. Hence all these SSE surviving the intuitive criterion are ruled out by D1.

Clearly if $x_1^* = 1/2$ no deviation to the right is possible hence we conclude :

Proposition 8

The only SSE surviving D1 is for $\alpha \in ((9/\sqrt{5}) - 4 , 1/2]$:

$$\mathbf{T}_1^* = \mathbf{T}_1(0) \qquad x_1^* = 1/2$$

$$\mu^* = \frac{\pi_\alpha(1/2) - \pi_\alpha^0}{\pi_\alpha(1/2)[1 - \bar{\pi}_\alpha^0]} \qquad \pi^*(c - \alpha / x_1^*) = \bar{\pi}_\alpha(1/2) < 1$$

7. THE MUTUALLY EXCLUSIVE DOMAINS IN THE (π , α) - SPACE

7.1. The Cho and Sobel theorems

Cho and Sobel (1988) show that under some assumptions, there exists a unique D1 equilibrium in signaling games. They also show that in monotonic signaling games, criterion D1 is equivalent to the Universal Divinity criterion and to the Never a Weak Best Response (NWBR) criterion. The sequential spatial model we considered in this paper is a case of monotonic signaling game and moreover the assumptions of Cho and Sobel's first result above are satisfied here. This will allow us to draw a map in the (π , α)-space showing the respective (and mutually exclusive) domains of pure strategy separating and pooling equilibria and least mixing semi-separating equilibria satisfying criterion D1.

We first prove that our model is a case of monotonic signaling game. Cho and Sobel's definition of monotonicity (expressed in our notation) is as follows : a signaling game is monotonic iff for all $x_1 \in [0, 1/2]$ and all x_2 and $x_2' \in BR(\pi_\alpha, x_1) = \{x_2 / \exists\, \pi_\alpha : x_2 = \tilde{x}_2 (x_1 , \pi_\alpha)\}$ whenever one type of firm 1 prefers a situation (x_1 , x_2) to a situation (x_1 , x_2') then the other type of firm 1 does also. That is :

$$\{PR_1 (x_1 , x_2) > PR_1 (x_1 , x_2')\} \Leftrightarrow \{PR_1^\alpha (x_1 , x_2) > PR_1^\alpha (x_1 , x_2')\}$$

Since $\dfrac{\partial PR_1}{\partial x_2} > 0$ and $\dfrac{\partial PR_1^\alpha}{\partial x_2} > 0$, either inequality implies that $x_2 > x_2'$

which implies the other inequality. Hence we have monotonicity which by Cho and Sobel's result justifies our approach of not considering Universal Divinity or NWBR criteria since they are for monotonic signaling game equivalent to D1.

To derive their result stating that there exists a unique D1-equilibrium outcome, Cho and Sobel make the following 5 assumptions when the set of signals (the possible locations of firm 1) is a compact interval and the set of reactions by the receiver (the possible locations of firm 2) is also a compact interval. Let us define $PR_1 (x_1 , x_2)$ as $PR_1 (0, x_1 , x_2)$ and $PR_1^\alpha (x_1 , x_2)$ as $PR_1(\alpha, x_1 , x_2)$ and similarly for PR_2 and PR_2^α ; let $T = \{\{t\} \in \{0, \alpha\}\}$; then Cho and Sobel's 5 assumptions can be stated as :

A1. Monotonicity (as defined above)

A2. $PR_2(t, x_1, x_2)$ is continuous in (x_1, x_2) and a strictly quasiconcave, differentiable function of x_2.

A3. $\partial PR_2(.) / \partial x_2$ is an increasing function of t

A4. If $t < t'$ and $x_1 < x_1'$, then
$PR_1(t, x_1, x_2) \leq PR_1(t, x_1', x_2')$ implies $PR_1(t', x_1, x_2) < PR_1(t', x_1', x_2')$.

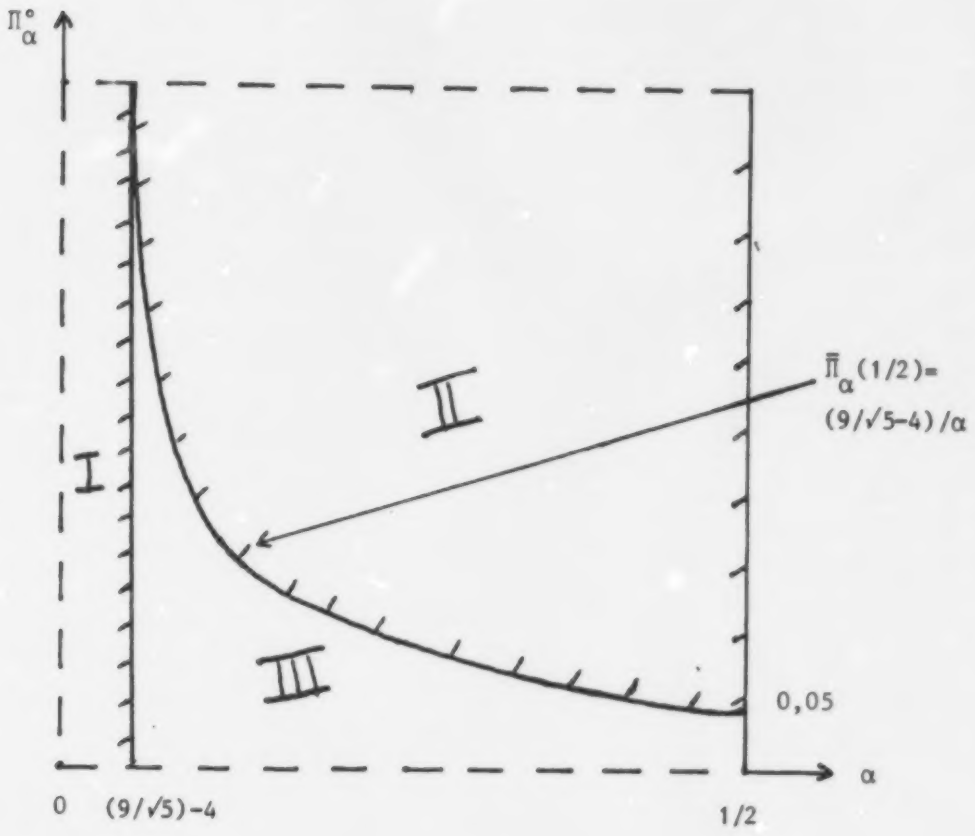
A5. For each t, $PR_1(t, x_1, x_2^*(t, x_1))$ is a strictly quasiconcave function of x_1.

We have already shown that A1 holds in our case. Using (2.2) and (6.2), it is straightforward to verify that A2 is satisfied. Noting that t increases from 0 (for the high cost type) to α (for the low cost type), we easily check that A3 is satisfied. Assumption A4 states that if the high cost firm 1 makes at least as much profits in situation (x_1', x_2') as in situation (x_1, x_2) with $x_1' > x_1$ (x_1' to the right of x_1) then the low cost firm 1 does strictly better in (x_1', x_2') than in (x_1, x_2). This assumption A4 is implied in our case of a unidimensional signal by a single-crossing condition stating that $-\dfrac{\partial PR_1 / \partial x_1}{\partial PR_1 / \partial x_2}$ is decreasing with t ; from (6.1), this expression is equal to $\dfrac{3x_1 - x_2 - t}{x_2 + x_1 + t}$ for $t \in [0, \alpha]$. Hence assumption A4 is satisfied [it is implied by the single-crossing condition and A1]. Finally, $PR_1(t, x_1, x_2^*(t, x_1))$ is obtained from (6.1) and (7) ; for both $t \in [0, \alpha]$, it is strictly concave in x_1. Hence A1 to A5 are verified : for each point $(\pi^\circ_\alpha, \alpha)$ in (π, α)-space, there is a unique D1-equilibrium outcome satisfying criterion D1 (and Universal Divinity and NWBR). Given the nature of the equilibria identified above and the implied outcomes, we conclude to the uniqueness of the equilibrium for each $(\pi^\circ_\alpha, \alpha)$.

7.2 The Map

From sections 4.3, 5.3 and 6.3, we know that for a $(\pi^\circ_\alpha, \alpha)$ such that $\alpha \in [0, (9/\sqrt{5}) - 4]$ and $\pi^\circ_\alpha \in [0, 1]$, the only equilibrium surviving D1 is the separating equilibrium most favorable to firm 1 (and also to firm 2) ; this is region I in figure 11. For a $(\pi^\circ_\alpha, \alpha)$ such that $\alpha \in ((9/\sqrt{5}) - 4, 1/2]$ and $\pi^\circ_\alpha \in [\bar{\pi}_\alpha(1/2), 1]$, the only equilibrium surviving D1 is the pooling equilibrium $(1/2, \pi^\circ_\alpha)$; this is region II in figure 11. Finally, for $(\pi^\circ_\alpha, \alpha)$ such that $\alpha \in ((9/\sqrt{5}) - 4, 1/2]$ and $\pi^\circ_\alpha \in [0, \bar{\pi}_\alpha(1/2))$, the only equilibrium satisfying D1 is the semi-separating

equilibrium with the high cost firm 1 randomizing between $\bar{x}_1^* = \bar{x}_1(0)$ and $x_1^* = 1/2$ (where the low cost firm 1 locates) with the probability of $x_1^* = 1/2$ being $\mu^* = \dfrac{\bar{\pi}_\alpha(1/2) - \pi^\circ_\alpha}{\bar{\pi}_\alpha(1/2)\,[1 - \pi^\circ_\alpha]} < 1$ and therefore $\pi^*(c - \alpha / x_1^*) = \bar{\pi}_\alpha(1/2)$; we observe that μ^* goes from 0 to 1 as π°_α goes from $\bar{\pi}_\alpha(1/2)$ to 0. Hence, the semi-separating equilibrium surviving D1 exists for $\pi^\circ_\alpha \in [0, \bar{\pi}_\alpha(1/2))$. Moreover, from (4) and (14) we have $\bar{\pi}_\alpha(1/2) = [(9/\sqrt{5}) - 4] / \alpha$. This corresponds to region III in figure 11.



I | III The frontier between I and III belongs to I

THE MAP OF D1 EQUILIBRIA

FIGURE 11

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FOOTNOTES

(1) Our analysis could be generalized to other forms of price competition. But it is well known that, for example, under mill pricing Nash equilibrium does not exist in pure strategies (cf d'Aspremont, Jaskold - Gabszewicz and Thisse (1979)).

(2) For the case of sunk costs see our companion paper Boyer, Laffont, Mahenc and Moreaux (1990).

(3) In a circular market with an uniform density of identical consumers and transportation costs depending only on the distance between the consumer and the firm, the choice of location by firm cannot transmit any information about its marginal cost. Hence the linear segment with uniform density of identical consumers is the simplest spatial structure permitting signaling.

APPENDIX I

For $\alpha \in (0 , (9 / \sqrt{5}) - 4)$, $\bar{x}_1^{(-)}$ and $\bar{x}_1^{(+)}$ are the roots of the following equation :

$$\overline{PR}_1 (x_1 , 1) - \overline{PR}_1 (\bar{x}_1(0) , 0) = 0$$

For $\alpha \in [1/6, 1/2]$, note that $\underline{PR}_1(x_1(0), 0) = \underline{PR}_1(1/2, 0)$; therefore the expression of $\underline{x}_1^{(-)}$ changes :

$$\underline{PR}_1 (x_1 , 1) - \underline{PR}_1 (1/2, 0) = 0$$

$$\Leftrightarrow - 20 x_1^2 + 16 (1 + 2 \alpha) x_1 + 7 \alpha^2 - 8 \alpha - 3 = 0$$

$$x_1^{(-)} = (4 (1+ 2 \alpha) - \Delta'^{1/2}/10$$

where $\Delta' = 99 \alpha^2 + 24 \alpha + 1$

i.e :

$$- 5 x_1^2 + 2 (2 + \alpha) x_1 + [(\alpha + 2)^2/4] - (9/5) = 0$$

hence :

$$\bar{x}_1^{(-)} = (2 + \alpha - \bar{\Delta}^{1/2}) / 5$$

$$\bar{x}_1^{(+)} = (2 + \alpha + \bar{\Delta}^{1/2}) / 5$$

where $\bar{\Delta} = 9[(\alpha^2 / 4) + \alpha]$

For $\alpha \in (0, 26 - (26^2 - 1)^{1/2})$, $\underline{x}_1^{(-)}$ and $\underline{x}_1^{(+)}$ are the roots of the following equation :

$$\underline{PR}_1 (x_1 , 1) - \underline{PR}_1 (\underline{x}_1(0), 0) = 0$$

i.e :

$$- 5 x_1^2 + 4 (1 + 2\alpha) x_1 + (2\alpha + 1)^2 - 9(2 + 3\alpha)^2 /20 = 0$$

hence :

$$\underline{x}_1^{(-)} = (2 (1 + 2\alpha) - \underline{\Delta}^{1/2}) / 5$$

$$\underline{x}_1^{(+)} = (2 (1 + 2\alpha) + \underline{\Delta}^{1/2}) / 5$$

where :

$$\underline{\Delta} = (63 \alpha^2 / 4) + 9\alpha$$

For $\alpha \in [1/6, 1/2]$, note that $\underline{PR}_1(\underline{x}_1(0), 0) = \underline{PR}_1(1/2, 0)$; therefore the expression

of $\underline{x}_1^{(-)}$ changes :

$$\underline{PR}_1(x_1, 1) - \underline{PR}_1(1/2, 0) = 0 \iff -20\, x_1^2 + 16(1+2\alpha)\, x_1 + 7\,\alpha^2 - 8\,\alpha - 3 = 0$$

$$\underline{x}_1^{(-)} = (4\,(1+2\alpha) - \underline{\Delta}'^{1/2}) / 10$$

where :

$$\underline{\Delta}' = 99\,\alpha^2 + 24\,\alpha + 1$$

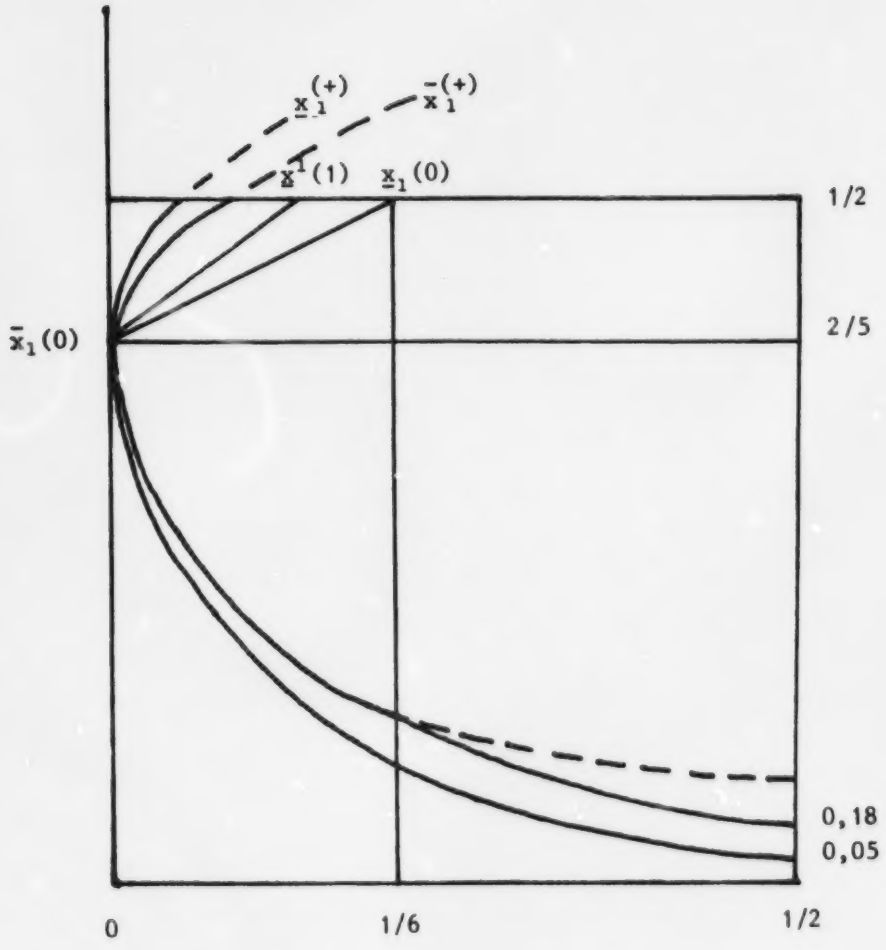

FIGURE A.I.1

Note that :

$$\forall\, \alpha \in (0,\, 1/2]$$

$$\bar{x}_1^{(-)} < \underline{x}_1^{(-)} < \bar{x}_1^{(+)} < \underline{x}_1^{(+)}$$

APPENDIX II

Let us define $\tilde{\pi}^{\alpha}(x_1, \dot{x}_1, \dot{\pi}_{\alpha})$ and $\underset{\sim}{\pi}^{\alpha}(x_1, \dot{x}_1, \dot{\pi}_{\alpha})$ as the greatest root of (A.II.1) and (A.II.2) respectively :

$$\overline{PR}_1 (x_1, \pi_{\alpha}) = \overline{PR}_1 (\dot{x}_1, \dot{\pi}_{\alpha}) \qquad \text{(A.II.1)}$$

$$\underline{PR}_1 (x_1, \pi_{\alpha}) = \underline{PR}_1 (\dot{x}_1, \dot{\pi}_{\alpha}) \qquad \text{(A.II.2)}$$

Note that (A.II.1) and (A.II.2) are given by :

$$(1/36)\,[\,\alpha^2\,\pi_{\alpha}^{\,2} + 2\pi_{\alpha}(4\alpha x_1 + 2\alpha) - 20x_1^2 + 16x_1 + 4] - \overline{PR}(\dot{x}_1, \dot{\pi}_{\alpha}) = 0 \qquad \text{(A.II. 1 bis)}$$

$$(1/36)\,[\,\alpha^2\,\pi_{\alpha}^{\,2} + 2\pi_{\alpha}(4\alpha x_1 + 2\alpha + 3\alpha^2) - 20x_1^2 + 8x_1(2 + 3\alpha) + 4 + 12\alpha + 9\alpha^2\,] - \underline{PR}(\dot{x}_1, \dot{\pi}_{\alpha}) = 0 \qquad \text{(A.II. 2 bis)}$$

The sum of the roots of (A.II.1 bis) and (A.II.2 bis) are :

$$-2(4x_1 + 2 + 3\alpha)/\alpha < 0$$

and

$$-4(2x_1 + 1)/\alpha < 0$$

so (A.II.1) and (A.II.2) have only one positive root if any.

Totally differentiating (A.II.1) and (A.II.2)

$$\frac{\partial \tilde{\pi}^{\alpha}(x_1, \dot{x}_1, \dot{\pi}_{\alpha})}{\partial x_1} = -\frac{\partial \overline{PR}_1 / \partial x_1}{\partial \overline{PR}_1 / \partial \pi_{\alpha}} = -\frac{-40x_1 + 8(\alpha\,\pi_{\alpha} + 2)}{2\alpha^2\,\pi_{\alpha} + 2(4\alpha x_1 + 2\alpha)}$$

$$\frac{\partial \pi^{\alpha}(x_1, \dot{x}_1, \dot{\pi}_{\alpha})}{\partial x_1} = -\frac{\partial \underline{PR}_1 / \partial x_1}{\partial \underline{PR}_1 / \partial x_1} = -\frac{-40x_1 + 8(\alpha\,\pi_{\alpha} + 3\alpha + 2)}{2\alpha^2\,\pi_{\alpha} + 2(4\alpha x_1 + 2\alpha + 3\alpha^2)}$$

so that :

$$\frac{\partial \tilde{\pi}^{\alpha}}{\partial x_1} = \frac{\partial \pi^{\alpha}}{\partial x_1} + \frac{2^2\ 3^2\ x_1}{(\alpha\ \pi_{\alpha} + 4x_1 + 1)\ (\alpha\ (\pi_{\alpha} + 3) + 4x_1)}$$

hence :

$$\forall\ (x_1\ ,\ \dot{x}_1) \in [0\ ,\ 1/2]^2 \quad ,\ \forall\ \dot{\pi}_{\alpha} \in [0\ ,\ 1]$$

$$\frac{\partial \tilde{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})}{\partial x_1} > \frac{\partial \underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})}{\partial x_1} \qquad \text{(A.II.3)}$$

Remember now that for $\bar{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$ and $\underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$ as defined in section 4.2., we have :

$$\bar{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha}) = \min\ \{\tilde{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})\ ,\ 1\ \} \qquad \text{(A.II.4)}$$

$$\underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha}) = \min\ \{\underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})\ ,\ 1\ \} \qquad \text{(A.II.5)}$$

and that :

$$x_1 \in [\ \bar{x}_1^{(-)}(\dot{x}_1\ ,\ \dot{\pi}_{\alpha})\ ,\ \bar{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_{\alpha})] \qquad \text{(A.II.6)}$$

$$\Rightarrow\ \bar{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha}) = \tilde{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$$

$$x_1 \in [\ \underset{\sim}{x}_1^{(-)}(\dot{x}_1\ ,\ \dot{\pi}_{\alpha})\ ,\ \underset{\sim}{x}_1^{(+)}(\dot{x}_1\ ,\ \dot{\pi}_{\alpha})] \qquad \text{(A.II.7)}$$

$$\Rightarrow\ \underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha}) = \underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$$

From (A.II.3) to (A.II.7) we derive the respective behavior of $\bar{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$ and $\underset{\sim}{\pi}^{\alpha}(x_1\ ,\ \dot{x}_1\ ,\ \dot{\pi}_{\alpha})$ according to the locations of $\dot{x}_1$, $\bar{x}_1\ (\dot{\pi}_{\alpha})$ and $\underset{\sim}{x}_1\ (\dot{\pi}_{\alpha})$

Case : $\dot{x}_1 < \bar{x}_1\ (\dot{\pi}_{\alpha})$

$$\Rightarrow \left.\frac{\partial \underset{\sim}{\pi}^{\alpha}(.)}{\partial x_1}\right|_{x = \dot{x}_1} < 0 \quad \text{and} \quad \left.\frac{\partial \bar{\pi}^{\alpha}(.)}{\partial x_1}\right|_{x_1 = \dot{x}_1} < 0$$

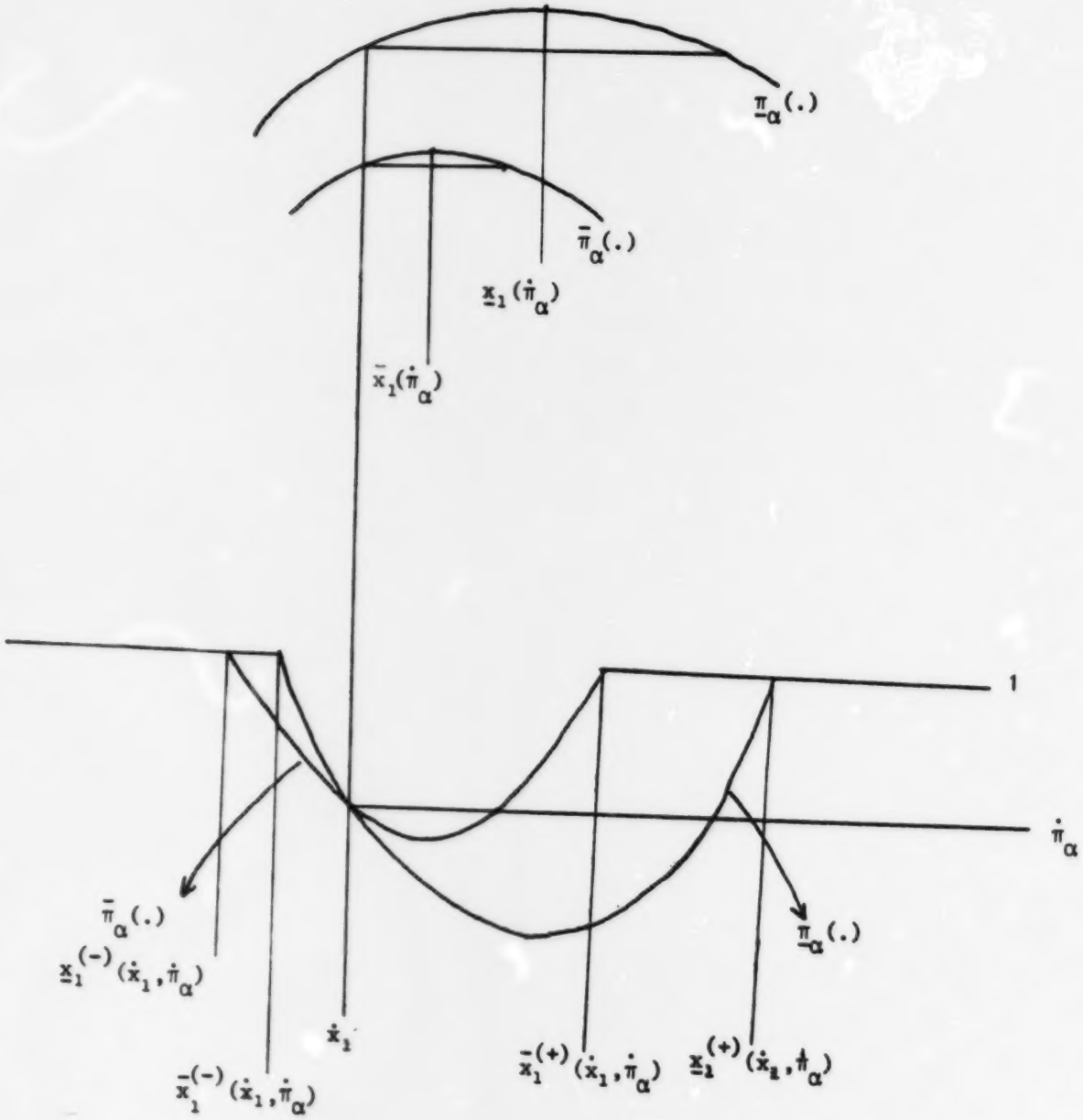
$\underline{\pi}_\alpha(.)$
$\bar{\pi}_\alpha(.)$
$\underline{x}_1(\dot{\pi}_\alpha)$
$\bar{x}_1(\dot{\pi}_\alpha)$
1
$\dot{\pi}_\alpha$
$\bar{\pi}_\alpha(.)$
$\underline{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha)$
$\dot{x}_1$
$\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha)$
$\bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$
$\underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$
$\underline{\pi}_\alpha(.)$

FIGURE A.II.1

Case : $\bar{x}_1(\dot{\pi}_\alpha) \ll \dot{x}_1 < \underline{x}_1(\dot{\pi}_\alpha)$

$$\Rightarrow \quad \frac{\partial \underline{\pi}^{\alpha}(.)}{\partial x_1}\Bigg|_{x = \dot{x}_1} < 0 \quad \text{and} \quad \frac{\partial \bar{\pi}^{\alpha}(.)}{\partial x_1}\Bigg|_{x_1 = \dot{x}_1} \geqslant 0$$

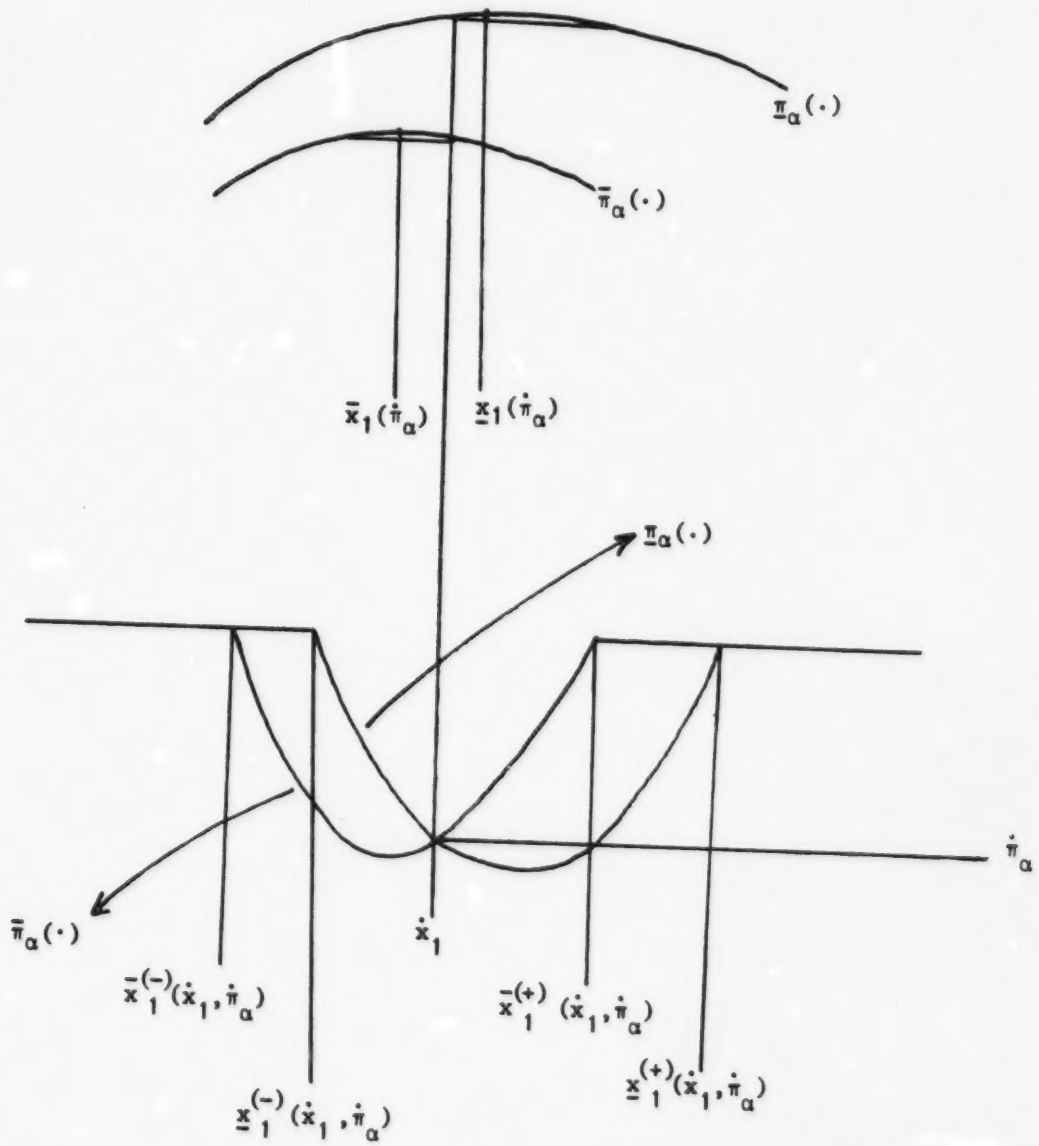

FIGURE A.II.2

Case : $\underline{x}_1(\dot{\pi}_\alpha) \lessdot \dot{x}_1$

$$\Rightarrow \quad \frac{\partial \bar{\pi}^\alpha(.)}{\partial x_1}\Big|_{x = \dot{x}_1} > 0 \quad \text{and} \quad \frac{\partial \underline{\pi}^\alpha(.)}{\partial x_1}\Big|_{x_1 = \dot{x}_1} \geqslant 0$$

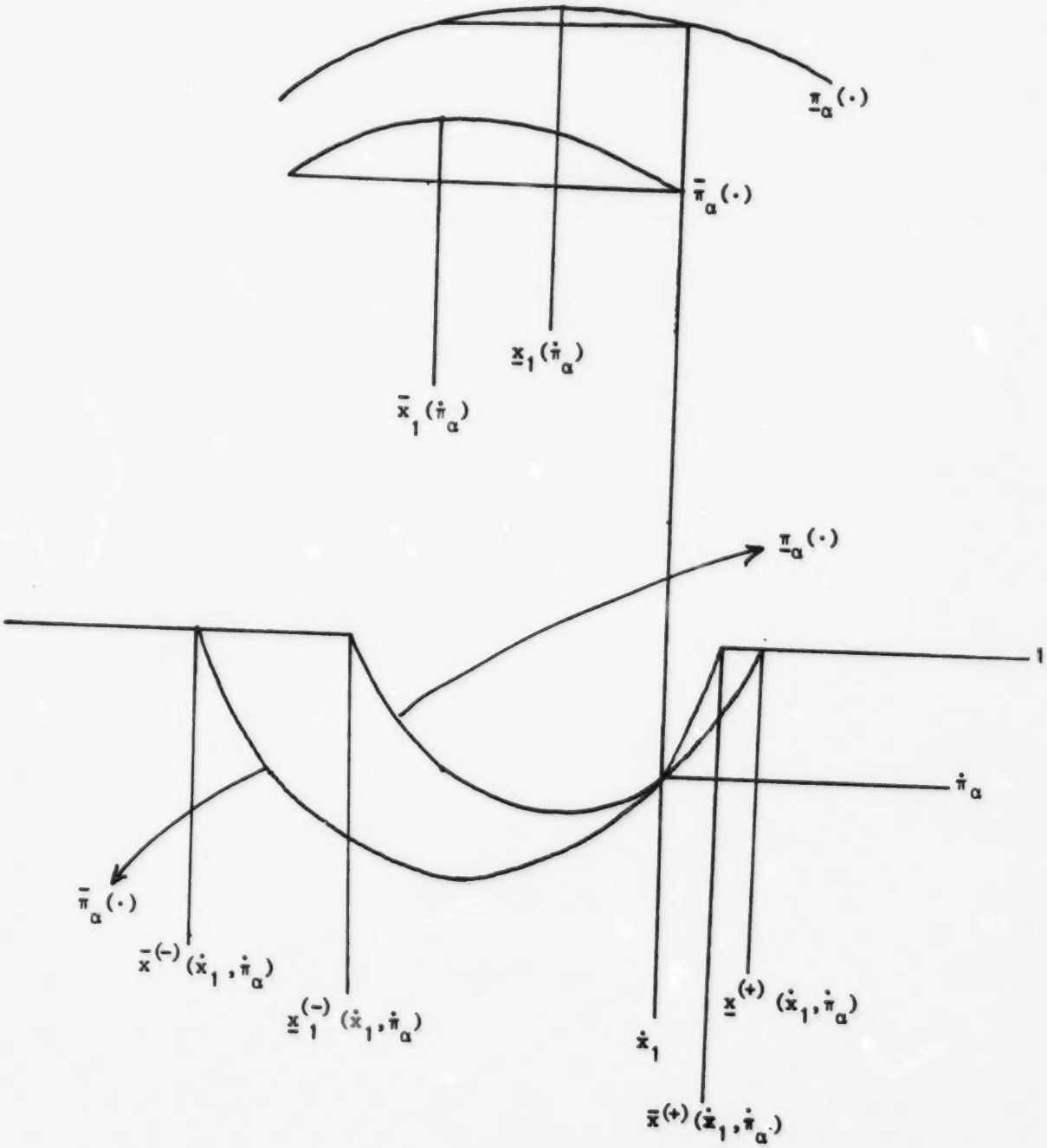

FIGURE A.II.3

We derive from the former study that :

$$\forall\, \dot{x}_1 \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}]$$

$$\forall\, \dot{\pi}_\alpha \geqslant \max\ \{\bar{\pi}_\alpha(\dot{x}_1), \underline{\pi}_\alpha(\dot{x}_1)\}$$

$$\bar{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha) \leqslant \underline{x}_1^{(-)}(\dot{x}_1, \dot{\pi}_\alpha) \leqslant \bar{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha) \leqslant \underline{x}_1^{(+)}(\dot{x}_1, \dot{\pi}_\alpha)$$

APPENDIX III

Determination of the frontiers between zones A, B, C and D (figure 10)

Remember that for any firm 2's prior belief $\pi^0{}_\alpha$ and any Bayesian posterior belief function $\pi^*(\ /x_1)$ which support x_1^* as a pooling equilibrium

$$\pi^*(c - \alpha / x_1^*) = \pi^0{}_\alpha \geqslant \max \{\bar{\pi}_\alpha(x_1^*), \underline{\pi}_\alpha(x_1^*)\}$$

For $\alpha \in (0, 1/6]$ we are looking for $x_1 = \dot{x}_1^\alpha$ such that $\bar{\pi}_\alpha(x_1) = \underline{\pi}_\alpha(x_1)$, i.e solution of the following system :

$$\begin{cases} \overline{PR}_1(x_1, \bar{\pi}_\alpha(x_1)) - \overline{PR}_1(\bar{x}_1(0), 0) = 0 \\ \underline{PR}_1(x_1, \underline{\pi}_\alpha(x_1)) - \underline{PR}_1(\underline{x}_1(0), 0) = 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} -20 x_1^2 + 8(2 + \alpha \bar{\pi}_\alpha(x_1) + 3\alpha) x_1 \\ \qquad + (\alpha \underline{\pi}_\alpha(x_1))^2 + 4\alpha \underline{\pi}_\alpha(x_1) + 6\alpha^2 \underline{\pi}_\alpha(x_1) \\ \qquad - (36\alpha^2/5) - (48\alpha/5) - (16/5) = 0 \\ -20 x_1^2 + 8(2 + \alpha \bar{\pi}_\alpha(x_1)) x_1 + \alpha^2 \bar{\pi}_\alpha^2(x_1) + 4\alpha \bar{\pi}_\alpha(x_1) \\ \qquad - 16/5 = 0 \end{cases}$$

Such a solution exists and is unique for $\alpha \in [0, 1/6]$

Note that for $\alpha \in [1/6, 1/2]$ the expression of $\dot{x}_1^\alpha$ is the following :

$\dot{x}_1^\alpha = (7920\alpha^2 - 240\alpha + 1) / 180\alpha$.

Then we are looking for $x_1 = \tilde{x}_1^{\alpha}$ such that $\bar{\pi}^{\alpha}(x_1, 1/2, 1) = \Pi_{\alpha}(x_1)$, i.e solution of the following system :

$$\begin{cases} \overline{PR}_1 (x_1, \Pi_{\alpha}(x_1)) - \overline{PR}_1 (1/2, 1) = 0 \\ \underline{PR}_1 (x_1, \Pi_{\alpha}(x_1)) - \underline{PR}_1 (\underline{x}_1(0), 0) = 0 \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} -20 x_1^2 + 8 (2 + \alpha \Pi_{\alpha}(x_1)) x_1 \\ \qquad + (\alpha \Pi_{\alpha}(x_1) + 2)^2 - 7 - 8\alpha - \alpha^2 = 0 \\ -20 x_1^2 + 8 (2 + \alpha \Pi_{\alpha}(x_1) + 3\alpha) x_1 + (\alpha \Pi_{\alpha}(x_1))^2 \\ \qquad + 6\alpha^2 \Pi_{\alpha}(x_1) - 16 \alpha^2 / 5 + 4 \alpha \Pi_{\alpha}(x_1) \\ \qquad - 48 \alpha / 5 - 16 / 5 = 0 \end{cases}$$

Such a solution exists and is unique for $\alpha \in [\frac{9}{\sqrt{5}} - 4, \alpha^k]$ where $\alpha^k = (243^{1/2} - 14) / 47 (\simeq 0{,}034)$.

For the case $\alpha = (9\sqrt{5}) - 4$, $\bar{\pi}^{\alpha}(x_1, 1/2, 1) = \bar{\pi}_{\alpha}(x_1)$ for each $x_1 \in [\underline{x}_1^{(-)}, \bar{x}_1^{(+)}]$ (where $\bar{x}_1^{(+)} = 1/2$), so that :

$$X_{1\alpha}^{+} = [\underline{x}_1^{(-)}, 1/2]$$

$$X_{1\alpha}^{-} = \emptyset$$

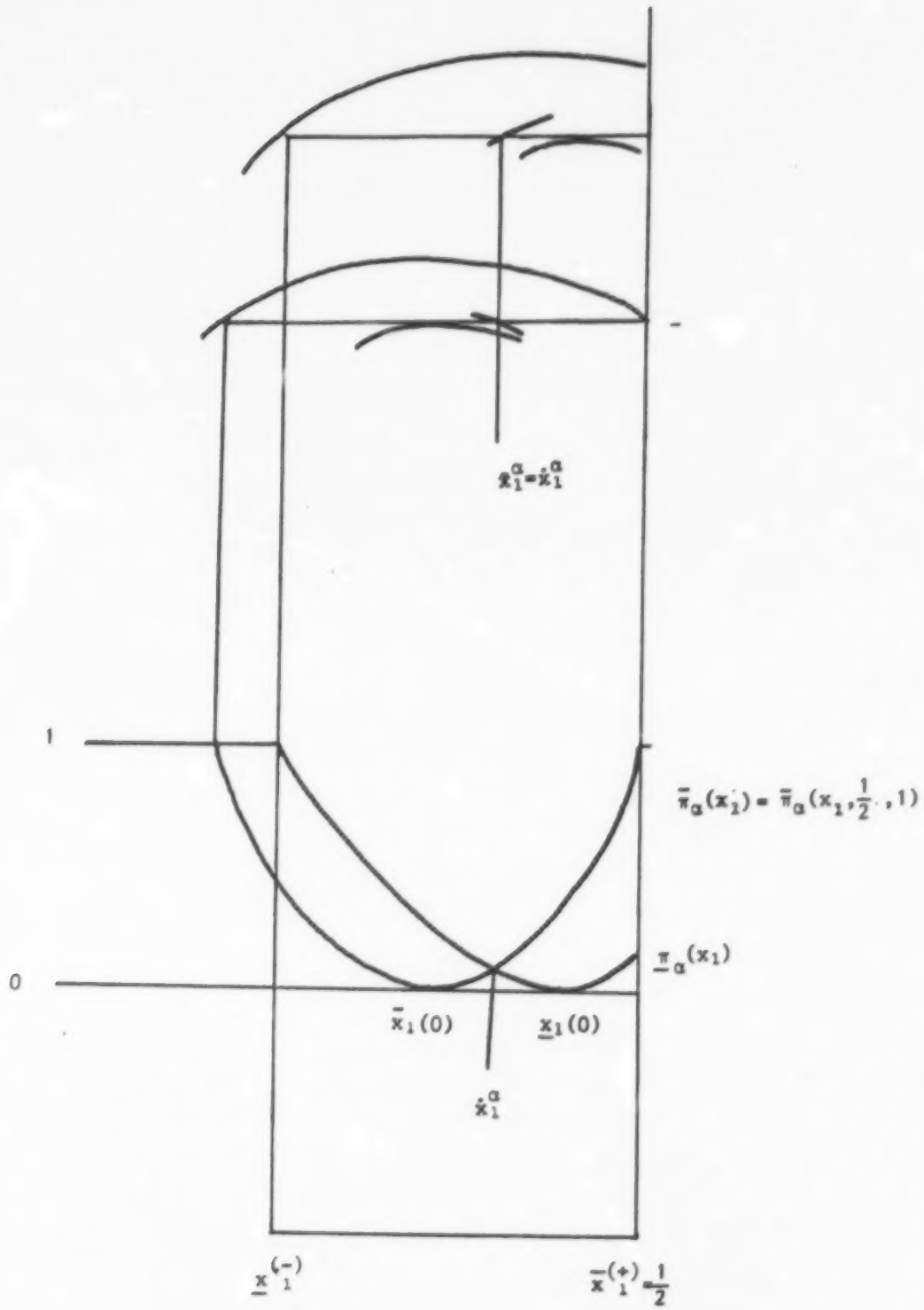

FIGURE A.III.1

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\dot{x}_1^{\alpha}, 1/2)$ and $\pi^*(c - \alpha / x_1^*) \in [\bar{\pi}_{\alpha}(x_1^*, 1/2, 1), 1]$ is defining a pooling which is eliminated by K.I.C. since it is open to the leader to deviate to 1/2.

$[\dot{x}_1^{\alpha}, 1/2) \subset$ zone D (Figure 9).

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\underline{x}_1^{(-)}, \dot{x}_1^{\alpha})$ and $\pi^*(c - \alpha / x_1^*) \in [\underline{\pi}_{\alpha}(x_1^*), 1]$ is defining a pooling that is not surviving K.I.C.

$[\underline{x}_1^{(-)}, \dot{x}_1^{\alpha}) \subset$ zone C (Figure 9).

For the case $(9\sqrt{5}) - 4 < \alpha < \alpha^*$, $\tilde{x}_1^{\alpha} < \dot{x}_1^{\alpha}$, so that

$$X_{1\alpha}^{-} = [\underline{x}_1^{(-)}, \tilde{x}_1^{\alpha}]$$

$$X_{1\alpha}^{+} = [\tilde{x}_1^{\alpha}, 1/2]$$

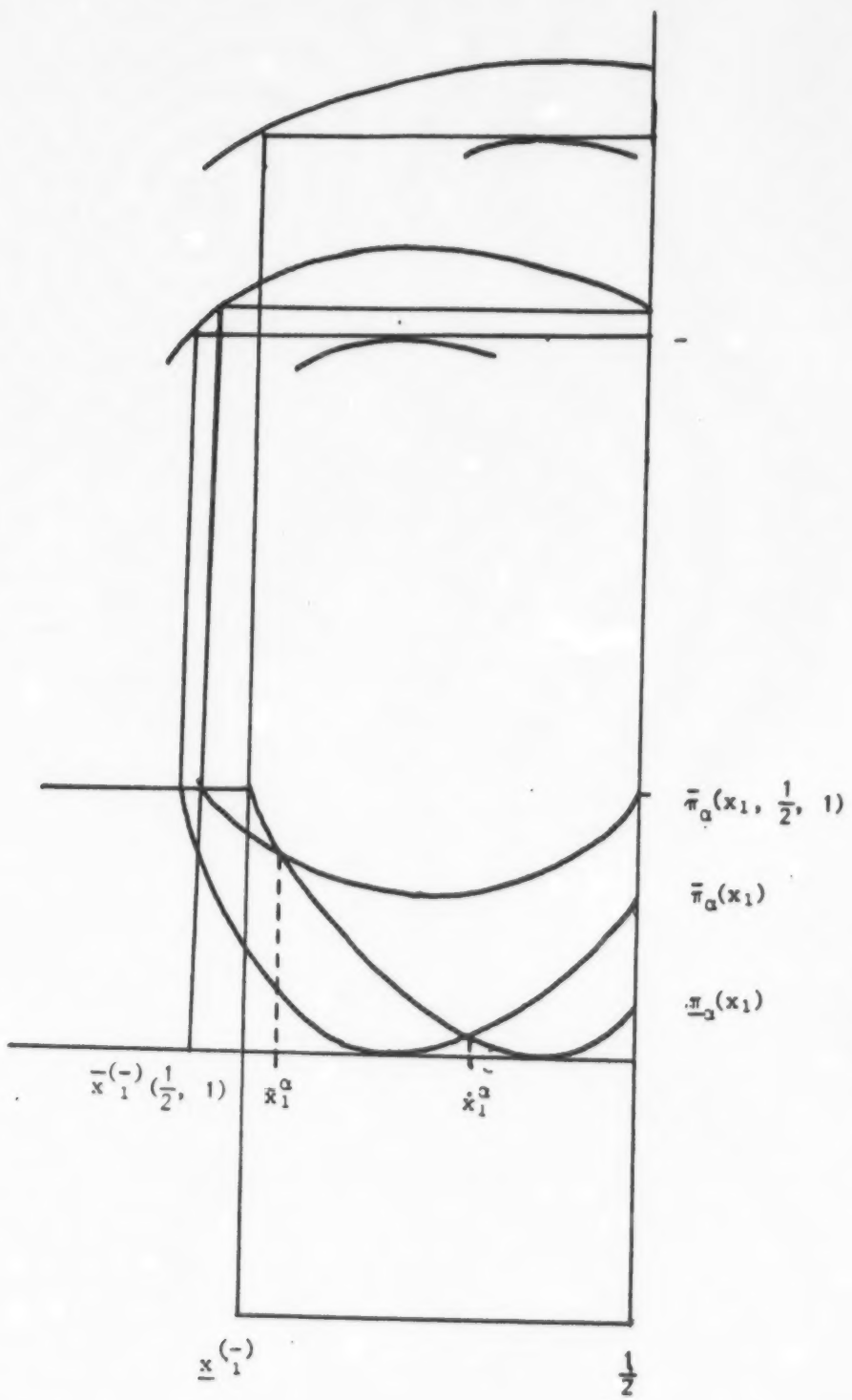

FIGURE A.III.2

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\tilde{x}_1^{\alpha}, 1/2)$ and $\pi^*(c - \alpha / x_1^*) \in [\bar{\pi}_{\alpha}(x_1^*, 1/2, 1), 1]$ is defining a pooling that is not surviving KIC.

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\tilde{x}_1^{\alpha}, 1/2)$ and $\pi^*(c - \alpha / x_1^*) \in [\max(\pi_{\alpha}(x_1^*), \bar{\pi}_{\alpha}(x_1^*)), \bar{\pi}_{\alpha}(x_1^*, 1/2, 1)]$ is defining a pooling that is surviving KIC.

$[\tilde{x}_1^{\alpha}, 1/2) \subset$ zone B (Figure 9).

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\underline{x}_1^{(-)}, \tilde{x}_1^{\alpha})$ and $\pi^*(c - \alpha / x_1^*) \in [\max(\pi_{\alpha}(x_1^*), 1]$ is defining a pooling that is not surviving KIC.

$[\underline{x}_1^{(-)}, x_1^{\alpha}) \subset$ zone C (Figure 9).

For the case $\alpha = \alpha^k$:

$$\bar{\pi}^{\alpha}(\underline{x}_1^{(-)}, 1/2, 1) = \bar{\pi}^{\alpha}(1/2)$$

$$\bar{\pi}^{\alpha}(x_1, 1/2, 1) > \bar{\pi}^{\alpha}(x_1) \qquad \forall x_1 \in (\underline{x}_1^{(-)}, 1/2]$$

hence :

$$X_{1\alpha}^{-} = \underline{x}_1^{(-)}$$

$$X_{1\alpha}^{+} = (\underline{x}_1^{(-)}, 1/2]$$

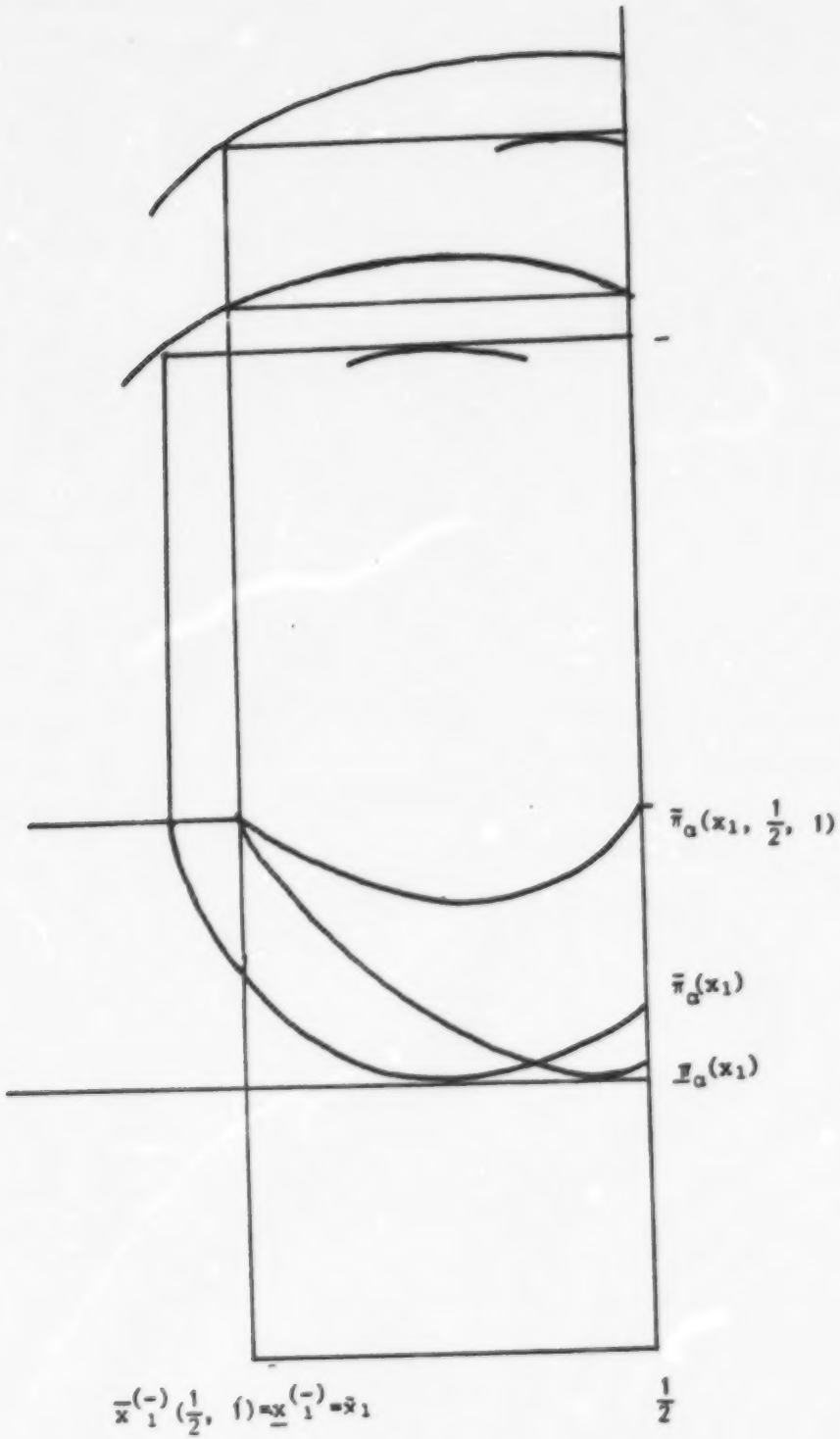

FIGURE A.III.3

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\underline{x}_1^{(-)}, 1/2)$ and $\pi^*(c - \alpha / x_1^*) \in [\max(\bar{\pi}_\alpha(x_1^*, 1/2), 1]$ is defining a pooling that is not surviving K.I.C.

Any $(x_1^*, \pi^*(c - \alpha / x_1^*))$ for which $x_1^* \in [\underline{x}_1^{(-)}, 1/2)$ and $\pi^*(c - \alpha / x_1^*) \in [\max(\bar{\pi}_\alpha(x_1^*), \underline{\pi}_\alpha(x_1^*)), \bar{\pi}_\alpha(x_1^*, 1/2), 1)]$ is defining a pooling that is surviving K.I.C.

$[\underline{x}_1^{(-)}, 1/2) \subset$ zone B (Figure 9).

APPENDIX IV

Let $\dot{x}_1^{\alpha}$ be the unique solution of the following system :

$$\begin{cases} \overline{PR}_1 (x_1 , \bar{\pi}_{\alpha}(x_1)) - \overline{PR}_1(\bar{x}_1(0), 0) = 0 \\ \underline{PR}_1 (x_1 , \underline{\pi}_{\alpha}(x_1)) - \underline{PR}_1(\underline{x}_1(0), 0) = 0 \end{cases}$$

we have already seen that :

For $\alpha \in (0, 1/6]$ $\quad \dot{x}_1^{\alpha} = (\alpha^2 + 6\alpha + 4)^{1/2}/5$

For $\alpha \in [1/6, 1/2]$ $\quad \dot{x}_1^{\alpha} = (7920\ \alpha^2 - 240\ \alpha + 1)^{1/2}/180\ \alpha$

which is illustrated on the following figures :

For $\alpha \in (0, 1/6)$:

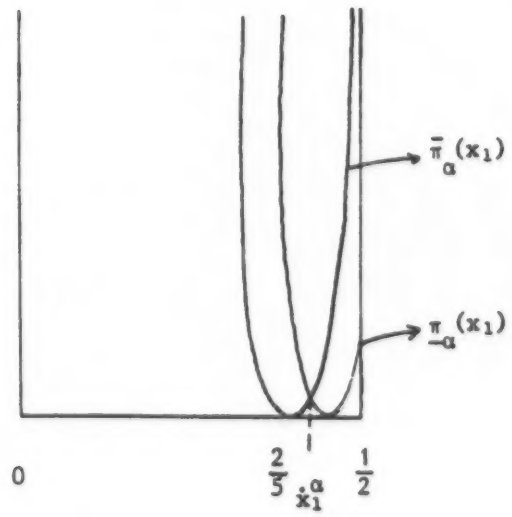

FIGURE A.IV.1

For $\alpha \geq 1/6$:

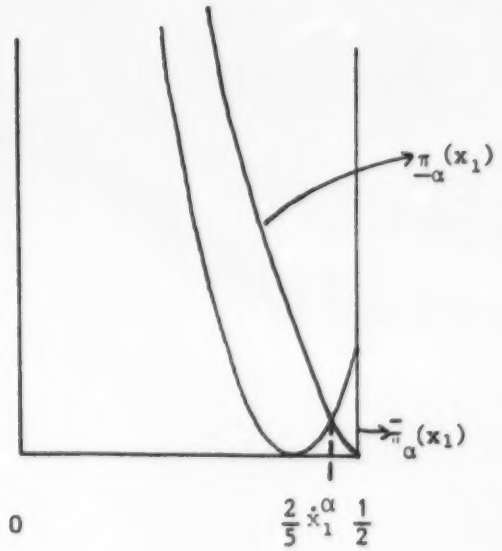

FIGURE A.IV.2

APPENDIX V

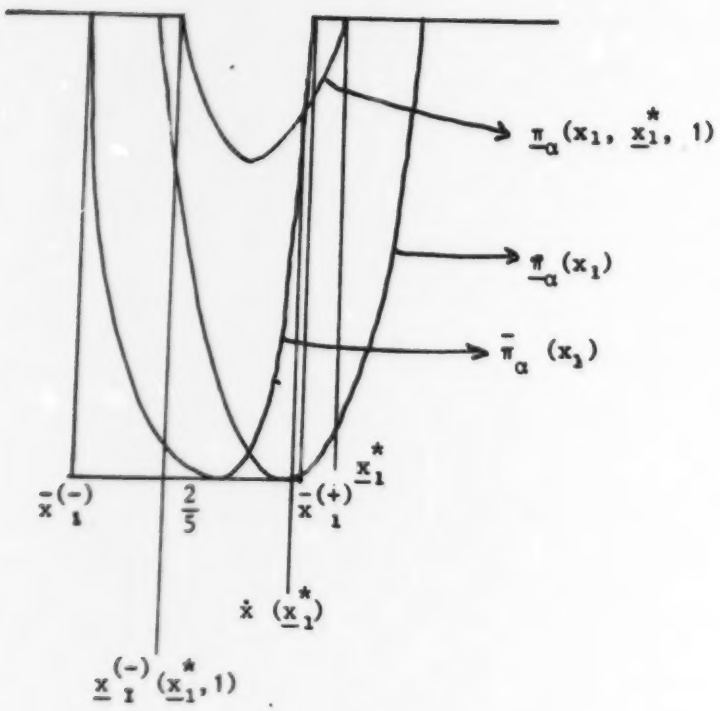

FIGURE A.V.1